PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 119, Number 1, September 1993

ALMOST ISOMETRIC COPIES OF l_∞ IN SOME BANACH SPACES

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. It is shown that any σ-complete Banach lattice, with an order semi-continuous norm containing an isomorphic copy of l_∞, contains an almost isometric copy of l_∞. It is also proved that any Fenchel-Orlicz space (resp. the subspace of finite elements of any Fenchel-Orlicz space) generated by an Orlicz function not satisfying the suitable Δ_2-condition contains an almost isometric copy of l_∞ (resp. c_0).

1. INTRODUCTION

Two Banach spaces X, Y are said to be $(1+\varepsilon)$-*isometric* provided there exists a linear isomorphism $T: X \to Y$ such that $\|T\|\,\|T^{-1}\| \le 1+\varepsilon$. By scaling we can arrange that T is a $(1+\varepsilon)$-isometry if

$$\|x\|_X \le \|Tx\|_Y \le (1+\varepsilon)\|x\|_X$$

for any $x \in X$. We say that a Banach space X contains an *almost isometric copy* of Y if for any $\varepsilon > 0$ there exists a subspace Z in X such that Z, Y are $(1+\varepsilon)$-isometric.

Note that Krivine [Kr] proved that if a Banach space X contains l_p^n's $(1+\varepsilon)$-uniformly for some $\varepsilon > 0$, $1 \le p \le \infty$, then it also contains them almost isometrically. For $p = 1$ or $p = \infty$, this result goes back to James and for $p = 2$ to Dvoretzky. Let us recall that the well-known result of James [J] shows that a Banach space X contains an almost isometric copy of c_0 (or l_1) whenever it contains an isomorphic copy of c_0 (resp. l_1).

In this paper we show a similar result for l_∞-copies; however, we restrict ourselves only to special Banach spaces.

Note also that, as far as we know, for $1 < p < \infty$ it is unknown whether or not any Banach space isomorphic to l_p contains a subspace almost isometric to l_p. This is known as the "distortion problem" [LP].

In the sequel X denotes a Banach space; $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_+^e$ stand for the reals, nonnegative reals, and extended (by $+\infty$) nonnegative reals. In what follows (Ω, Σ, μ) denotes an arbitrary σ-finite measure space. For the sake of simplicity we will consider only nonatomic and purely atomic (the counting)

Received by the editors October 9, 1991 and, in revised form, January 29, 1992.
1991 *Mathematics Subject Classification*. Primary 46B03, 46B42, 46E30.

measure. $L^0(\mu, X)$ denotes the space of all (equivalence classes of) strongly Σ-measurable functions defined on Ω with values in X.

A map $\Phi : X \to \mathbb{R}_+$ is said to be an *Orlicz function* if $\Phi(0) = 0$, Φ is continuous at 0, lower semicontinuous on X, even, and

$$(*) \qquad \Phi \text{ is bounded on any ball in } X,$$

$$(**) \qquad \inf\{\Phi(x) : \|x\| = r\} \to \infty \quad \text{as} \quad r \to \infty.$$

Given an arbitrary Orlicz function Φ we define a functional $I_\Phi : L^0(\mu, X) \to \mathbb{R}_+^e$ by

$$I_\Phi(f) = \int_\Omega \Phi(f(t))\, d\mu,$$

which is even and convex, $I_\Phi(0) = 0$, and for any $f \in L^0(\mu, X)$ the condition $I_\Phi(\lambda f) = 0$ for any $\lambda > 0$ yields $f = 0$.

The *Fenchel-Orlicz* space $L^\Phi(\mu, X)$ generated by an Orlicz function Φ is defined as the set of all functions $f \in L^0(\mu, X)$ such that $I_\Phi(\lambda f) < \infty$ for some $\lambda > 0$ depending on f (cf. [T2] and in the scalar case also [KR, Lu, M]).

The subspace $E^\Phi(\mu, X)$ of $L^\Phi(\mu, X)$ (called the subspace of finite elements) is defined by

$$E^\Phi(\mu, X) = \{f \in L^\Phi(\mu, X) : I_\Phi(\lambda f) < \infty \text{ for any } \lambda > 0\}.$$

The spaces L^Φ and E^Φ can be equipped with the *Luxemburg norm*

$$\|f\| = \inf\{\varepsilon > 0 : I_\Phi(f/\varepsilon) \le 1\}$$

as well as with the *Orlicz norm*

$$\|f\|^0 = \sup\left\{\int_\Omega \langle f(t), g(t)\rangle\, d\mu : g \in L^0(\mu, X^*),\ I_{\Phi^*}(f) \le 1\right\},$$

where X^* denotes the dual space of X and Φ^* is the *complementary function* to Φ in the sense of Young, i.e.,

$$\Phi^*(x^*) = \sup\{\langle x, x^*\rangle - \Phi(x) : x \in X\}$$

for any $x^* \in X^*$. It is well known (cf. [N] and in the scalar case also [KR, RR]) that

$$\|f\|^0 = \inf\{k^{-1}(1 + I_\Phi(kf)) : k > 0\}.$$

Furthermore, $(L^\Phi(\mu, X), \|\cdot\|)$ is a Banach space (cf. [T2] and in the scalar case also [KR, Lu, M]).

We say that an Orlicz function Φ satisfies the Δ_2-*condition* for all $x \in X$ (at infinity) [at zero] if there are positive constants K and C such that $\Phi(2x) \le K\Phi(x)$ for all $x \in X$ (for $x \in X$ satisfying $\Phi(x) \ge c$) [for $x \in X$ satisfying $\Phi(x) \le c$].

An Orlicz function Φ satisfies the suitable Δ_2-condition if it satisfies the Δ_2-condition for all $x \in X$ when μ is nonatomic infinite, the Δ_2-condition at infinity if μ is nonatomic finite, and the Δ_2-condition at zero if μ is the counting measure (cf. [RR]).

It is known (cf. [H] and in the scalar case also [K, T1, T2]) that any Fenchel-Orlicz space $L^\Phi(\mu, X)$, with the Luxemburg norm, containing an isomorphic copy of l_∞ contains also an isometric copy of l_∞.

However, the Orlicz space $(L^\Phi(\mu, \mathbb{R}), \|\cdot\|^0)$ need not contain an isometric copy of l_∞ whenever it contains an isomorphic copy of l_∞ (this follows from the criteria for rotundities of $L^\Phi(\mu, \mathbb{R})$ equipped with the Orlicz norm (cf. [T1, RR]). It will be proved in this paper that in the case of the Orlicz norm the Fenchel-Orlicz space $L^\Phi(\mu, X)$ contains an almost isometric copy of l_∞ whenever it contains an isomorphic copy of l_∞.

Recall that a Banach lattice X is said to be *σ-complete* if every order bounded sequence in X has a supremum. A σ-complete Banach lattice X is said to have an *order continuous norm* (an *order semicontinuous norm*) whenever $x_n \downarrow 0$ implies $\|x_n\| \to 0$ (resp. $0 \le x_n \uparrow x$, $x \in X$, implies $\|x_n\| \to \|x\|$).

2. Results

We start with the following

Theorem 1. *Let E be a σ-complete Banach lattice with an order semicontinuous norm. If E contains an isomorphic copy of l_∞, then it contains an almost isometric copy of l_∞.*

Proof. From the well-known result of Lozanovskii [L] it follows that the norm in E is not order continuous whenever E contains an isomorphic copy of l_∞. Since E is σ-complete (by the assumption), in virtue of the result of Ando [A] (cf. also [KA, p. 382; LT, p. 7]) it follows that there exists an order bounded sequence (u_k) of mutually disjoint positive elements in E satisfying $c = \inf_n \|u_n\| > 0$.

Assume that $0 \le u_n \le x \in X$ holds for all $n \in \mathbb{N}$ and put

$$K_n = \sup\left\{\left\|\sum_{i=n}^{m} u_i\right\| : m \in \mathbb{N}\right\}$$

for $n \in \mathbb{N}$. Since (K_n) is a nonincreasing sequence satisfying $c \le K_n \le \|x\|$, it is convergent and $c \le K = \lim_{n\to\infty} K_n \le \|x\|$.

Let $0 < \varepsilon < 1$ be fixed. Take $0 < \theta < 1 < \eta$ such that $\theta/\eta \ge 1 - \varepsilon$. Now pick k_1 in such a way that $K_{k_1} < \eta K$. It follows from the definition of K_n that for a certain $k_2 > k_1$, we get

$$\left\|\sum_{i=k_1}^{k_2-1} u_i\right\| > \theta K_{k_1} \ge \theta K.$$

We can construct, by the induction, an increasing sequence (k_n) in $\mathbb{N}$ such that

$$\left\|\sum_{i=k_n}^{k_{n+1}-1} u_i\right\| > \theta K, \tag{1}$$

$$K_{k_n} < \eta K. \tag{2}$$

Now consider the sequence (x_n) of positive and pairwise disjoint elements, where

$$x_n = \sum_{i=k_n}^{k_{n+1}-1} (\eta K)^{-1} u_i.$$

For each $0 \leq \xi = (\xi_n) \in l_\infty$, we put

$$T(\xi) = \sup\left\{\sum_{i=1}^{n} \xi_i x_i : n \in \mathbb{N}\right\}.$$

The supremum exists because E is σ-order continuous and

$$\sum_{i=1}^{n} \xi_i x_i \leq (\eta K)^{-1} x \|\xi\|_\infty.$$

Since $T : l_\infty^+ \to E^+$ and T is linear, it extends uniquely to a linear operator $\overline{T}$ from l_∞ into E.

If $\xi = (\xi_n) \in l_\infty$, then the order semicontinuity of the norm and (2) yield

$$\begin{aligned}\|\overline{T}(\xi)\| &\leq \|\overline{T}(|\xi|)\| = \|T(|\xi|)\| \\ &= \lim_{n\to\infty}\left\|\sum_{i=1}^{n} |\xi_i| x_i\right\| \leq \|\xi\|_\infty \sup_n \left\|\sum_{i=1}^{n} x_i\right\| \\ &\leq (\eta K)^{-1} K_{k_1} \|\xi\|_\infty \leq \|\xi\|_\infty.\end{aligned}$$

Furthermore, by (1)

$$\|T\xi\| \geq |\xi_n|\,\|x_n\| \geq \theta\eta^{-1}|\xi_n| \geq (1-\varepsilon)|\xi_n|$$

for all $n \in \mathbb{N}$. Consequently,

$$(1-\varepsilon)\|\xi\|_\infty \leq \|T\xi\| \leq \|\xi\|_\infty$$

for any $\xi \in l_\infty$. This finishes the proof.

Note that Fenchel-Orlicz spaces are not isomorphic to Banach lattices in general. Because of this we are interested in the problem of almost isometric copies of l_∞ as well as c_0 in these spaces. In order to present our results we first need to prove some auxiliary lemmas.

Lemma 1. *If an Orlicz function Φ does not satisfy the suitable Δ_2-condition, then there exists a sequence (f_n) of functions with pairwise disjoint supports and such that $\|f_n\|^0 = 1$ for any $n \in \mathbb{N}$ and $I_\Phi(f_n) \to 0$ as $n \to \infty$.*

Proof. It is known (cf. [H, K, T1]) that if Φ does not satisfy the suitable Δ_2-condition, then there exists a sequence (g_n) in $L^\Phi(\mu, X)$ with pairwise disjoint supports and such that $I_\Phi(g_n) \to 0$ and $\|g_n\| = 1$ for any $n \in \mathbb{N}$. We have $\|g_n\|^0 \geq 1$, whence defining $f_n = g_n/\|g_n\|^0$ we obtain $\|f_n\|^0 = 1$ and $0 \leq I_\Phi(f_n) \leq I_\Phi(g_n) \to 0$, i.e., $I_\Phi(f_n) \to 0$.

Lemma 2. *For any $f \in L^\Phi(\mu, X)$ and $\delta > 0$ we have $\|f\|^0 \leq 1 + \delta$ whenever $I_\Phi(f) \leq \delta$.*

Proof. Under the assumptions we have

$$\|f\|^0 = \inf_{k>0} k^{-1}(1 + I_\Phi(kf)) \leq 1 + I_\Phi(f) \leq 1 + \delta.$$

Theorem 2. *If Φ is an Orlicz function on X not satisfying the suitable Δ_2-condition, then for any $\varepsilon > 0$ there exists a $(1+\varepsilon)$-isometry T of l_∞ into $(L^\Phi(\mu, X), \|\cdot\|^0)$ such that $Tc_0 \not\subset E^\Phi(\mu, X)$.*

Proof. Take an arbitrary $\varepsilon > 0$. Let (f_n) be the sequence in $L^\Phi(\mu, X)$ with pairwise disjoint supports such that

$$I_\Phi(f_n) \leq 2^{-n}\varepsilon \quad \text{and} \quad \|f_n\|^0 = 1$$

for any $n \in \mathbb{N}$, which is built as in the proof of Lemma 1. Define an operator $T : l_\infty \to L^\Phi(\mu, X)$ by $T\xi = \sum_{n=1}^\infty \xi_n f_n$ for any $\xi = (\xi_n) \in l_\infty$. We have by the orthogonal additivity of I_Φ

$$I_\Phi(T\xi/\|\xi\|_\infty) = \sum_{n=1}^\infty I_\Phi(\xi_n f_n/\|\xi\|_\infty) \le \sum_{n=1}^\infty I_\Phi(f_n) \le \varepsilon .$$

Thus, from Lemma 2 it follows that

$$\|T\xi\|^0 \le (1+\varepsilon)\|\xi\|_\infty$$

for every $\xi \in l_\infty$. Since $\|T\xi\|^0 \ge \|\xi_n f_n\|^0 = |\xi_n|$ for any $n \in \mathbb{N}$, we have

$$\|T\xi\|^0 \ge \|\xi\|_\infty .$$

This means that T is an $(1+\varepsilon)$-isometry. Note that $Te_n = f_n$, where e_n denotes the sequence of real numbers whose nth term is one and the rest are zero. We have $f_n = g_n/\|g_n\|^0$, where $I_\Phi(g_n) \to 0$ as $n \to \infty$ and $\|g_n\| = 1$ for $n \in \mathbb{N}$. Hence $I_\Phi(\lambda g_n) = \infty$ for any $\lambda > 1$ and $n \in \mathbb{N}$ large enough. Thus, it follows that $f_n \notin E^\Phi(\mu, X)$ for $n \in \mathbb{N}$ large enough. This finishes the proof.

Theorem 3. *If Φ is an Orlicz function on X not satisfying the suitable Δ_2-condition, then for any $\varepsilon > 0$ there exists an operator $T : l_\infty \to L^\Phi(\mu, X)$, which is a $(1+\varepsilon)$-isometry in the case of the Luxemburg norm as well as in the case of the Orlicz norm. Furthermore, T restricted to c_0 acts $(1+\varepsilon)$-isometrically to $E(\mu, X)$ with respect to both norms in $E(\mu, X)$.*

Proof. Note that the Δ_2-condition is equivalent to the Δ_l-condition for any $l > 1$. Note also that in view of conditions $(*)$ and $(**)$ an Orlicz function Φ satisfies the suitable Δ_2-condition for some constant $c > 0$ if and only if it satisfies this condition for an arbitrary constant $c > 0$. Therefore, if Φ does not satisfy the Δ_2-condition at infinity, then for any $\varepsilon > 0$ we can choose a sequence (x_n) in X and a sequence (A_n) in Σ such that

$$\Phi((1+\varepsilon)x_n) > 2^{n+1}\varepsilon^{-1}\Phi(x_n), \tag{3}$$

$$2^{-n-1}\varepsilon < \Phi(x_n)\mu(A_n) \le 2^{-n}\varepsilon \tag{4}$$

for any $n \in \mathbb{N}$. Note that in the case of a nonatomic measure we can get (4) in the form $\Phi(x_n)\mu(A_n) = 2^{-n}\varepsilon$.

Now, let T be an operator defined by

$$T\xi = \sum_{n=1}^\infty \xi_n x_n \chi_{A_n}$$

for any $\xi = (\xi_n)$ in l_∞.

If $\xi = (\xi_n) \in l_\infty$, then

$$I_\Phi(T\xi/\|\xi\|_\infty) \le \sum_{n=1}^\infty \Phi(x_n)\mu(A_n) \le \varepsilon ,$$

i.e., $T\xi \in L^\Phi(\mu, X)$.

Taking any $\xi = (\xi_n) \in c_0$ and any $\lambda > 0$ we can choose $m \in \mathbb{N}$ such that $\lambda|\xi_n| \leq 1$ for $n \geq m$. Hence

$$\begin{aligned} I_\Phi(\lambda T\xi) &= \sum_{n=1}^{m-1} \Phi(\lambda\xi_n x_n)\mu(A_n) + \sum_{n=m}^{\infty} \Phi(\lambda\xi_n x_n)\mu(A_n) \\ &\leq \sum_{n=1}^{m-1} \Phi(\lambda\xi_n x_n)\mu(A_n) + \sum_{n=m}^{\infty} \Phi(x_n)\mu(A_n) \\ &\leq \sum_{n=1}^{m-1} \Phi(\lambda\xi_n x_n)\mu(A_n) + \varepsilon \sum_{n=m}^{\infty} 2^{-n} < \infty, \end{aligned}$$

i.e., $T\xi \in E^\Phi(\mu, X)$ for any $\xi \in c_0$.

Moreover, the right inequality in (4) yields

$$\begin{aligned} \|T\xi/\|\xi\|_\infty\|^0 &= \inf_{k>0} k^{-1}(1 + I_\Phi(kT\xi/\|\xi\|_\infty)) \\ &\leq 1 + I_\Phi(T\xi/\|\xi\|_\infty) \leq 1 + \varepsilon, \end{aligned}$$

whence

$$\|T\xi\| \leq \|T\xi\|^0 \leq (1+\varepsilon)\|\xi\|_\infty$$

for any $\xi \in l_\infty$.

On the other hand, for $0 < \varepsilon < 1$ and some $n \in \mathbb{N}$, applying (3) and the left inequality in (4), we get

$$\begin{aligned} I_\Phi((1+2\varepsilon)T\xi/\|\xi\|_\infty) &\geq \Phi((1+\varepsilon)x_n)\mu(A_n) \\ &> 2^{n+1}\varepsilon^{-1}\Phi(x_n)\mu(A_n) > 1, \end{aligned}$$

which yields

$$\|T\xi\|^0 \geq \|T\xi\| \geq (1+2\varepsilon)^{-1}\|\xi\|_\infty$$

for any $\xi \in l_\infty$. This finishes the proof.

Added in Proof. Recently, E. Odell and T. Schlumprecht have answered the distortion problem negatively.

Acknowledgment

The authors thank the referee for valuable remarks.

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