

THE SPECIAL VALUES AT NEGATIVE INTEGERS OF DIRICHLET SERIES ASSOCIATED WITH POLYNOMIALS OF SEVERAL VARIABLES

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ABSTRACT. Let $P(X)$ be a product of k linear forms in r variables X_1, \dots, X_r as given by

$$P(X_1, \dots, X_r) = \prod_{j=1}^k (a_{j1}X_1 + \dots + a_{jr}X_r + \delta_j),$$

$$\operatorname{Re} a_{ji} > 0, \operatorname{Re} \left(\delta_j + \sum_{i=1}^r a_{ji} \right) > 0.$$

Suppose that $\beta = (\beta_1, \dots, \beta_r)$ is an r -tuple of nonnegative integers. Consider the zeta function

$$Z(P, \beta)(s) = \sum_{n_1=1}^{\infty} \dots \sum_{n_r=1}^{\infty} n_1^{\beta_1} \dots n_r^{\beta_r} P(n_1, \dots, n_r)^{-s}, \quad \operatorname{Re} s > \frac{r + |\beta|}{k},$$

where $|\beta| = \beta_1 + \dots + \beta_r$. $Z(P, \beta)(s)$ has an analytic continuation in the whole complex plane and it is regular at $s = 0, -1, -2, \dots, -m, \dots$. In this paper, we shall compute the explicit values of $Z(P, \beta)(s)$ at $s = 0, -1, -2, \dots, -m, \dots$ and express them in terms of finite sums of polynomials in Bernoulli numbers.

1. INTRODUCTION AND NOTATION

Let $P(X_1, \dots, X_r)$ be a polynomial of r variables with positive coefficients in the real number field \mathbf{R} and $\xi = (\xi_1, \dots, \xi_r) \in \mathbf{C}^n$ with $|\xi_i| = 1$, $\xi_i \neq 1$. In [1], Cassou-Nogués considered the zeta function $Z(P, \xi)(s)$ associated with P and ξ as follows:

$$Z(P, \xi)(s) = \sum_{n \in \mathbf{N}^r} P(n)^{-s} \xi^n = \sum_{n_1=1}^{\infty} \dots \sum_{n_r=1}^{\infty} P(n_1, \dots, n_r)^{-s} \xi_1^{n_1} \dots \xi_r^{n_r}.$$

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She obtained among other results that the special values of the continuation of $Z(P, \xi)(s)$ at nonpositive integers is given by

$$Z(P, \xi)(-m) = R(P^m)(\xi).$$

Here

$$R(P^m)(T) = \sum_{n \in \mathbb{N}^r} P^m(n) T^n$$

is a formal power series. Also in [1], $R(P^m)(\xi)$ is expressed as a finite sum of numbers of the form $\lambda(P^m)/(1 - \xi_1)^{\alpha_1} \cdots (1 - \xi_r)^{\alpha_r}$, where α_i is a nonnegative integer and $|\alpha| > 0$. Obviously, the formula for $Z(P, \xi)(-m)$ did not work for the case when some $\xi_i = 1$.

Let $Z(P)(s) = \sum_{n \in \mathbb{N}^r} P(n)^{-s}$. In an effort to get a similar formula for $Z(P)(-m)$, Cassou-Nogués in [2] used two different ways to investigate this problem for certain kinds of polynomials. In particular, she considered those polynomials P with $P(X_1, \dots, X_r) = C_1 X_1^{m_1} + \cdots + C_r X_r^{m_r} + R(X_1, \dots, X_r)$, where $R(X)$ is a combination of monomials $X_1^{\alpha_1} \cdots X_r^{\alpha_r}$ satisfying $\sum_{i=1}^r \frac{\alpha_i}{m_i} \leq 1$. However no explicit formula for $Z(P)(-m)$ was obtained except for the special case $r = 2$ and $P(X, Y) = X^{m_1} + Y^{m_1} + X^{\alpha_1} Y^{\alpha_2}$.

It is clear that the evaluation of $Z(P)(s)$ at nonpositive integers is equivalent to finding the asymptotic expansion of the series $\sum_{n \in \mathbb{N}^r} e^{-P(n)t}$ at $t = 0$. Fractional powers may appear in the expansion and it makes the whole process more complicated.

On the other hand, Shintani [12] decomposed the Dedekind zeta function ζ_K of a totally real number field K into a finite sum of zeta functions $Z(P)(s)$ as defined above with

$$P(X) = \prod_{j=1}^k L_j(X) = \prod_{j=1}^k (a_{j1} X_1 + \cdots + a_{jr} X_r + \delta_j), \quad a_{ji} > 0, \quad \delta_j > 0.$$

Let $\Gamma(s)$ be the gamma function defined by $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$, $\text{Re } s > 0$. Then for $\text{Re } s > r/k$, we have the integral expression

$$Z(P)(s) [\Gamma(s)]^k = \int_0^\infty \cdots \int_0^\infty \frac{(t_1 \cdots t_k)^{s-1} e^{-[D, T]} dt_1 \cdots dt_k}{(e^{[A_1, T]} - 1) \cdots (e^{[A_k, T]} - 1)}.$$

Here $[D, T] = \delta_1 t_1 + \cdots + \delta_r t_r$ and $[A_i, T] = a_{i1} t_1 + \cdots + a_{ik} t_k$.

With the above formula, Shintani was able to express $Z(P)(-m)$ in terms of coefficients of power expansions of certain functions. It turned out that these coefficients are products of Bernoulli polynomials. On the other hand, the author obtained an explicit formula for the values at nonpositive integers of a zeta function associated with a polynomial [3].

In this paper, we shall consider a more general kind of zeta function $Z(P, \beta)(s)$ defined by

$$Z(P, \beta)(s) = \sum_{n_1=1}^\infty \cdots \sum_{n_r=1}^\infty n_1^{\beta_1} \cdots n_r^{\beta_r} P(n_1, \dots, n_r)^{-s}, \quad \text{Re } s > \frac{r + |\beta|}{k}.$$

Here $\beta = (\beta_1, \dots, \beta_r)$ is an r -tuple of nonnegative integers and $P(X) = \prod_{j=1}^k L_j(X)$, $L_j(X) = (a_{j1} X_1 + \cdots + a_{jr} X_r + \delta_j)$, $\text{Re } a_{ji} > 0$, $\text{Re } L_j(n) > 0$ for $n \in \mathbb{N}^r$.

To state our main result, we let J^m be the linear extension of $C[X_1, \dots, X_r]$ to C satisfying

$$J^m(X_1^{\alpha_1} \cdots X_r^{\alpha_r}) = \prod_{i=1}^r \zeta(-\alpha_i) = (-1)^{|\alpha|} \prod_{i=1}^r \frac{B_{\alpha_i+1}}{(\alpha_i+1)}.$$

Here $B_0, B_1, B_2, \dots, B_p, \dots$ are Bernoulli numbers defined by

$$\frac{t}{e^t - 1} = \sum_{p=0}^{\infty} \frac{B_p t^p}{p!}, \quad |t| < 2\pi.$$

For $m = 0$, we let $J^0(c) = c$. We can now give our main theorem.

Main Theorem. *Let $Z(P, \beta)(s)$ be the zeta function defined as above. Then $Z(P, \beta)(s)$ is an analytic function of s for $\operatorname{Re} s > (r + |\beta|)/k$ and it has an analytic continuation to the whole complex plane which is analytic except possible poles at $s = j/k$, $j \leq (r + |\beta|)$, where s is a nonnegative integer or zero. Furthermore, the special value of $Z(P, \beta)(s)$ at nonpositive integers is given by*

$$Z(P, \beta)(-m) = \frac{1}{k} \sum_{j=1}^k \sum_{\{j_1, \dots, j_p\}} J^{r-p} \left[\int_{\Delta_j(X_{j_1}, \dots, X_{j_p})} X^\beta P^m(X) dX_{j_1} \cdots dX_{j_p} \right].$$

Here in the second summation, $\{j_1, \dots, j_p\}$ ranges over all subsets of $\{1, \dots, r\}$. The domain of integration $\Delta_j(X_{j_1}, \dots, X_{j_p})$ is realized as a p -simplex in \mathbf{R}^p defined by $L_j(X) \geq 0$, $X_{j_1} \leq 0, \dots, X_{j_p} \leq 0$, by assuming that the coefficients of $L_j(X)$ are positive real numbers (see Proposition 6 for the precise meaning of the integral).

Remark. When $L_j(X)$ has real coefficients, then $\Delta_j(X_{j_1}, \dots, X_{j_p})$ is indeed a p -simplex. When $p = 0$, the term $J^r[X^\beta P^m(X)]$ resulted easily from our calculation as we shall see.

Concerning the rationality of the values of $Z(P, \beta)(s)$ at nonpositive integers, we have the following corollary.

Corollary. *Suppose that \mathbf{k} is an algebraic number field over the rational number field \mathbf{Q} and P (as given before) is a polynomial function over \mathbf{k} . Then for any integer m with $m \geq 0$, $Z(P, \beta)(-m) \in \mathbf{k}$.*

2. THE INTEGRAL FORMULA OF $Z(P, \beta)(s)$

From now on, we use the following notation:

$$\begin{aligned} A_j &= [a_{1j}, \dots, a_{kj}], & D &= [\delta_1, \dots, \delta_k], & T &= [t_1, \dots, t_k], \\ U &= [u_1, \dots, u_k], & u_k &= 1 - u_1 - \cdots - u_{k-1}, & dU &= du_1 \cdots du_{k-1}, \\ [A_j, T] &= \sum_{i=1}^k a_{ij} t_i, & [A_j, U] &= \sum_{i=1}^k a_{ij} u_i, \\ [D, T] &= \sum_{i=1}^k \delta_i t_i, & [D, U] &= \sum_{i=1}^k \delta_i u_i, \\ |T| &= t_1 + \cdots + t_k, & L_j(X) &= a_{j1} X_1 + \cdots + a_{jr} X_r + \delta_j. \end{aligned}$$

Proposition 1. For $\operatorname{Re} s > |\beta| + r$, we have

$$\begin{aligned} Z(P, \beta)(s)\Gamma(s)^k &= \int_0^\infty \cdots \int_0^\infty (t_1 \cdots t_k)^{s-1} e^{-[D, T]} \\ &\quad \times \prod_{j=1}^r \left[\sum_{n_j=1}^\infty n_j^{\beta_j} e^{-[A_j, T]n_j} \right] dt_1 \cdots dt_k. \end{aligned}$$

Proof. For $\operatorname{Re} s > 0$ and positive integers n_1, \dots, n_r , we have

$$\begin{aligned} &n_1^{\beta_1} \cdots n_r^{\beta_r} \prod_{j=1}^k (n + a_{j1}n_1 + \cdots + a_{jr}n_r + \delta_j)^{-s} \Gamma(s)^k \\ &= \int_0^\infty \cdots \int_0^\infty (t_1 \cdots t_k)^{s-1} n_1^{\beta_1} \cdots n_r^{\beta_r} \\ &\quad \times \prod_{j=1}^k \exp\{-(a_{j1}n_1 + \cdots + a_{jr}n_r + \delta_j)t_j\} dt_1 \cdots dt_k. \end{aligned}$$

Our assertion then follows from the monotone convergence theorem with which we can perform a term-by-term integration.

Proposition 2. For $\operatorname{Re} s > (|\beta| + r)/k$, we have

$$\begin{aligned} Z(P, \beta)(s)\Gamma(s)^k &= \int_0^\infty t^{ks-1} dt \int_E (u_1 \cdots u_k)^{s-1} e^{-[D, U]t} \\ &\quad \times \prod_{j=1}^r \left[\sum_{n_j=1}^\infty n_j^{\beta_j} e^{-n_j[A_j, U]t} \right] dU. \end{aligned}$$

Here E is the standard simplex in \mathbf{R}^{k-1} defined by

$$\begin{cases} u_1, \dots, u_{k-1} \geq 0, \\ 1 - u_1 - \cdots - u_{k-1} \geq 0. \end{cases}$$

Proof. The proof follows from Proposition 1 and the following changing of variables which map $(\mathbf{R}^+)^k$ onto $(\mathbf{R}^+) \times E$:

$$\begin{aligned} t_1 &= tu_1, \dots, t_{k-1} = tu_{k-1}, & t_k &= tu_k; \\ (u_1, \dots, u_{k-1}) &\in E, & u_k &= 1 - \sum_{i=1}^{k-1} u_i. \end{aligned}$$

Our assertion is true for $\operatorname{Re} s > |\beta| + r$.

Note that the function

$$\prod_{j=1}^r \left[t^{\beta_j+1} \sum_{n_j=1}^\infty n_j^{\beta_j} e^{-[A_j, U]t} \right]$$

is a rapidly decreasing function of t when $0 \leq t < \infty$. Consequently, the integral on the right-hand side is convergent for $\operatorname{Re} s > (|\beta| + r)/k$. Thus the formula holds for $\operatorname{Re} s > (|\beta| + r)/k$.

Remark. For $0 < t < 2\pi$ and positive integer p , we have

$$\left(\frac{d}{dt}\right)^p \sum_{n=1}^{\infty} e^{-nt} = \left(\frac{d}{dt}\right)^p \left(\frac{1}{t} + \sum_{j=1}^{\infty} \frac{B_j t^{j-1}}{j!} \right).$$

With a term-by-term differentiation, we get

$$\sum_{n=1}^{\infty} n^p e^{-nt} = \frac{p!}{t^{p+1}} + \sum_{j=1}^{\infty} \frac{(-1)^p B_{j+p+1} t^j}{(j+p+1) \cdot j!}.$$

Hence the series $t^{p+1} \sum_{n=1}^{\infty} n^p e^{-nt}$ is a regular function of t at $t = 0$.

3. THE ANALYTIC CONTINUATION OF $Z(P, \beta)(s)$

Let

$$(1) \quad F(t; U) = t^{|\beta|+r} e^{-[D, U]t} \prod_{j=1}^r \left\{ \sum_{n_j=1}^{\infty} n_j^{\beta_j} e^{-n_j [A_j, U]t} \right\}$$

and

$$(2) \quad I(s, t) = \frac{1}{\Gamma(s)^{k-1}} \int_E (u_1 \cdots u_k)^{s-1} F(t; U) du_1 \cdots du_{k-1}.$$

Then we can rewrite $Z(P, \beta)(s)\Gamma(s)$ as

$$Z(P, \beta)(s)\Gamma(s) = \int_0^{\infty} t^{ks-|\beta|-r-1} I(s, t) dt.$$

$F(t; U)$ is a continuous function in u_1, \dots, u_{k-1} and a rapidly decreasing function in t , and the measure

$$\frac{\Gamma(ks)(u_1 \cdots u_k)^{s-1} du_1 \cdots du_{k-1}}{\Gamma(s)^k}$$

is a bounded measure on the standard simplex E (see the following Remark). Thus $I(s, t)$ is a holomorphic function in s and a rapidly decreasing function for $0 \leq t < \infty$. Hence $Z(P, \beta)(s)\Gamma(s)$ has only possible simple poles at $s = j/k$, $j \leq r + |\beta|$, $j \in \mathbf{Z}$, with \mathbf{Z} being the set of integers.

Remark. The measure $\Gamma(ks)(u_1 \cdots u_k)^{s-1} du_1 \cdots du_{k-1} / \Gamma(s)^k$ is the well-known classical Dirichlet measure. It is well defined for $\operatorname{Re} s > 0$ and has its analytic continuation for all s .

Proposition 3. Let the power expansion of $F(t; U)$ at $t = 0$ be $\sum_{i=0}^{\infty} t^i F_i(U)$. Then

$$Z(P, \beta)(-m) = \frac{(-1)^m m!}{k} G_{km+|\beta|+r}(-m),$$

with

$$G_i(s) = \frac{1}{\Gamma(s)^{k-1}} \int_E (u_1 \cdots u_k)^{s-1} F_i(U) du_1 \cdots du_{k-1}, \quad \operatorname{Re} s > 0.$$

Proof.

$$\begin{aligned}
 Z(P, \beta)(s)\Gamma(s) &= \int_0^1 t^{ks-|\beta|-r-1} I(s, t) dt + \int_1^\infty t^{ks-|\beta|-r-1} I(s, t) dt \\
 &= \int_0^1 t^{ks-|\beta|-r-1} \left\{ \sum_{i=0}^\infty t^i G_i(s) \right\} dt + \int_1^\infty t^{ks-|\beta|-r-1} I(s, t) dt \\
 &= \sum_{i=0}^\infty \frac{G_i(s)}{ks-|\beta|-r+i} + \int_1^\infty t^{ks-|\beta|-r-1} I(s, t) dt.
 \end{aligned}$$

The integral on the right-hand side is convergent for any s , so the right-hand side is a meromorphic function in s , which gives the analytic continuation of $Z(P, \beta)(s)\Gamma(s)$. For $s = -m$, the residue of the right-hand side is $\frac{1}{k} G_{km+|\beta|+r}(-m)$. But as $\Gamma(s)$ has a simple pole at $s = -m$, it forces $Z(P, \beta)(s)$ to be regular at $s = -m$ and

$$\frac{(-1)^m}{m!} Z(P, \beta)(-m) = \frac{1}{k} G_{km+|\beta|+r}(-m).$$

It follows that

$$Z(P, \beta)(-m) = \frac{(-1)^m m!}{k} G_{km+|\beta|+r}(-m).$$

4. THE EXPLICIT FORMULA

Applying the formula

$$(3) \quad \sum_{n=1}^\infty n^p e^{-nt} = \frac{p!}{t^{p+1}} + \sum_{j=1}^\infty \frac{(-1)^j B_{j+p+1} t^j}{(j+p+1) \cdot j!}, \quad 0 < t < 2\pi,$$

we get the following power series expansions for $0 \leq t < 2\pi$:

$$\begin{aligned}
 (\Sigma_1) \quad & t^{\beta_1+1} \sum_{n_1=1}^\infty n_1^{\beta_1} e^{-n_1[A_1, U]t} \\
 &= \frac{\beta_1!}{[A_1, U]^{\beta_1+1}} + \sum_{j=0}^\infty \frac{(-1)^{\beta_1} B_{j+\beta_1+1}[A_1, U]^j t^{j+\beta_1+1}}{(j+\beta_1+1) \cdot j!},
 \end{aligned}$$

.....

$$\begin{aligned}
 (\Sigma_r) \quad & t^{\beta_r+1} \sum_{n_r=1}^\infty n_r^{\beta_r} e^{-n_r[A_r, U]t} \\
 &= \frac{\beta_r!}{[A_r, U]^{\beta_r+1}} + \sum_{j=0}^\infty \frac{(-1)^{\beta_r} B_{j+\beta_r+1}[A_r, U]^j t^{j+\beta_r+1}}{(j+\beta_r+1) \cdot j!},
 \end{aligned}$$

$$(\Sigma_{r+1}) \quad e^{-[D, U]t} = \sum_{j=0}^\infty \frac{(-1)^j [D, U]^j t^j}{j!}.$$

$G_{km+|\beta|+r}(s)$ is an integration over E of the coefficient of $t^{km+|\beta|+r}$ in the product of series $(\Sigma_1), \dots, (\Sigma_r), (\Sigma_{r+1})$. So it is a finite linear combination of

integrals of the form

(4)

$$H_\alpha(s) = \frac{1}{\Gamma(s)^{k-1}} \int_E (u_1 \cdots u_k)^{s-1} [A_1, U]^{\alpha_1} \cdots [A_r, U]^{\alpha_r} [D, U]^{\alpha_{r+1}} du_1 \cdots du_{k-1}$$

with $\alpha_i \geq 0$ or $\alpha_i = -\beta_i - 1$ ($i = 1, \dots, r$), $\alpha_{r+1} \geq 0$, and $\sum_{i=1}^{r+1} |\alpha_i| = km$. More precisely, the general terms in $G_{km+|\beta|+r}(s)$ can be divided into the following two types of integrals:

$$(1) \quad \sum_{|\alpha|=km} (-1)^{|\beta|} \left[\prod_{i=1}^r \frac{B_{\alpha_i+\beta_i+1}}{(\alpha_i + \beta_i + 1)\alpha_i!} \right] \frac{(-1)^{\alpha_{r+1}}}{\alpha_{r+1}!} H_\alpha(s),$$

where $\alpha = (\alpha_1, \dots, \alpha_r, \alpha_{r+1})$ is an $(r+1)$ -tuple of nonnegative integers, and

$$(2) \quad \sum_{\alpha} \beta_1! \cdots \beta_q! (-1)^{\beta_{q+1} + \cdots + \beta_r} \left[\prod_{i=q+1}^r \frac{B_{\alpha_i+\beta_i+1}}{(\alpha_i + \beta_i + 1)\alpha_i!} \right] \frac{(-1)^{\alpha_{r+1}}}{\alpha_{r+1}!} \\ \times \frac{1}{\Gamma(s)^{k-1}} \int_E \frac{(u_1 \cdots u_k)^{s-1} [A_{q+1}, U]^{\alpha_{q+1}} \cdots [A_r, U]^{\alpha_r} [D, U]^{\alpha_{r+1}} dU}{[A_1, U]^{\beta_1+1} \cdots [A_q, U]^{\beta_q+1}}.$$

Here in the summation, $\alpha = (\alpha_{q+1}, \dots, \alpha_r, \alpha_{r+1})$ ranges over all $(r-q+1)$ -tuples of integers satisfying $|\alpha| = km + \beta_1 + \cdots + \beta_q + q$.

In our propositions which follow, we shall devote ourselves to the evaluation of $H_\alpha(s)$ at $s = -m$ for various α 's appearing in the summation of $G_{klm+|\beta|+r}$ and combine them to yield the Bernoulli numbers associated with polynomials as given in the main theorem.

Proposition 4. Let $H_\alpha(s)$ be the integral as defined in (4). Then

$$\frac{(-1)^m m!}{k} \sum_{0 \leq \alpha_1, \dots, \alpha_{r+1} \leq m; |\alpha|=km} (-1)^{|\beta|} \left[\prod_{i=1}^r \frac{B_{\alpha_i+\beta_i+1}}{(\alpha_i + \beta_i + 1)\alpha_i!} \right] \frac{(-1)^{\alpha_{r+1}}}{\alpha_{r+1}!} H_\alpha(-m) \\ = J'[X^\beta P^m(X)].$$

Proof.

$$H_\alpha(s) = \prod_{i=1}^r \sum_{\alpha_{1i} + \cdots + \alpha_{ki} = \alpha_i} \frac{\alpha_i! (a_{1i})^{\alpha_{1i}} \cdots (a_{ki})^{\alpha_{ki}}}{\alpha_{1i}! \cdots \alpha_{ki}!} \\ \times \sum_{\alpha_{1r+1} + \cdots + \alpha_{kr+1} = \alpha_{r+1}} \frac{\alpha_{r+1}! (\delta_1)^{\alpha_{1r+1}} \cdots (\delta_k)^{\alpha_{kr+1}}}{\alpha_{1r+1}! \cdots \alpha_{kr+1}!} A_\alpha(s)$$

with

$$A_\alpha(s) = \frac{1}{\Gamma(s)^{k-1}} \int_E u_1^{s+b_1-1} \cdots u_k^{s+b_k-1} du_1 \cdots du_{k-1}.$$

Here $b_1 = \alpha_{11} + \alpha_{12} + \cdots + \alpha_{1r+1}$, \dots , $b_k = \alpha_{k1} + \alpha_{k2} + \cdots + \alpha_{kr+1}$. Note that

$$A_\alpha(s) = \frac{\Gamma(s+b_1) \cdots \Gamma(s+b_k)}{\Gamma(s)^{k-1} \Gamma(k s + |\alpha|)}$$

and

$$A_\alpha(-m) = \begin{cases} (-1)^{(k-1)m} \frac{(m!)^{k-1} k(km - |\alpha|)!}{(m-b_1)! \cdots (m-b_k)!} & \text{if } 0 \leq b_1, \dots, b_k \leq m, \\ 0 & \text{otherwise.} \end{cases}$$

In our cases, we have $b_1 + \cdots + b_k = |\alpha| = km$. Hence it forces $b_1 = b_2 = \cdots = b_k = m$ when $A_\alpha(-m)$ is not zero. For such cases, we have $A_\alpha(-m) = (-1)^{km-m}(m!)^{k-1}$. Our assertion then follows from the multinomial expansion of $X^\beta P^m(X)$ and our definition of $J^r[X^\beta P^m(X)]$.

For the remaining cases, we have to find the value of $s = -m$ of

$$\frac{1}{\Gamma(s)^{k-1}} \int_E \frac{(u_1 \cdots u_k)^{s-1} [A_{q+1}, U]^{\alpha_{q+1}} \cdots [A_r, U]^{\alpha_r} [D, U]^{\alpha_{r+1}} dU}{[A_1, U]^{\beta_1+1} \cdots [A_q, U]^{\beta_q+1}}$$

which is a linear combination of

$$G(\beta; b)(s) = \frac{1}{\Gamma(s)^{k-1}} \int_E \frac{u_1^{s+b_1-1} \cdots u_k^{s+b_k-1} dU}{[A_1, U]^{\beta_1+1} \cdots [A_q, U]^{\beta_q+1}} \quad (1 \leq q \leq r),$$

with $b_1 + \cdots + b_k = \alpha_{q+1} + \cdots + \alpha_{r+1} = km + q + \beta_1 + \cdots + \beta_q$. First we observe the following facts concerning $G(\beta; b)$.

(1) The integral in $G(\beta; b)(s)$ as a function of s has a pole at $s = -m$ of order at most $k-1$ arising from integration around the vertices of E .

(2) If there are more than one b_j in $\{b_1, \dots, b_k\}$ which are greater than m , then the order of the pole at $s = -m$ for the integral will be less than $k-1$. By considering the fact that $\Gamma(s)^{k-1}$ always has a pole at $s = -m$ of order $k-1$, we conclude that $G(\beta; b)(-m) = 0$.

With the above considerations, it suffices to consider the case when $b_j \geq m+1$ for exactly one j . In the following proposition, we suppose that $0 \leq b_1, \dots, b_{k-1} \leq m$ and $b_k \geq m+1$. To determine the explicit value of $G(\beta; b)(-m)$, we need the usual notation of partial differential operators. Let

$$D_1 = \frac{\partial}{\partial u_1}, \dots, D_{k-1} = \frac{\partial}{\partial u_{k-1}}, \quad D^\alpha = D_1^{\alpha_1} \cdots D_{k-1}^{\alpha_{k-1}}.$$

Also $\alpha! = \alpha_1! \cdots \alpha_{k-1}!$ if $\alpha = (\alpha_1, \dots, \alpha_{k-1})$.

Proposition 5. For $b = (b_1, \dots, b_k)$ with $|b| = km + |\beta| + q$, $0 \leq b_1, \dots, b_{k-1} \leq m$, and $b_k \geq m+1$, we have

$$G(\beta; b)(-m) = \sum_{\alpha_1 + \cdots + \alpha_{q+1} = \mathbf{m} - \mathbf{b}} \left[\prod_{j=1}^{k-1} \prod_{i=1}^q \left(\frac{a_{ki} - a_{ji}}{a_{ki}} \right)^{\alpha_{ji}} \frac{1}{\alpha_{ji}!} \right] \\ \times \prod_{i=1}^q \frac{(\beta_i + |\alpha_i|)! (-1)^{|\alpha_{q+1}|} (b_k - m - 1)!}{\beta_i! a_{ki}^{\beta_i+1} (b_k - m - 1 - |\alpha_{q+1}|)!}.$$

Here $\alpha_i = (\alpha_{1i}, \dots, \alpha_{k-1,i})$ ($i = 1, \dots, q+1$) and $\mathbf{m} - \mathbf{b} = (m - b_1, \dots, m - b_{k-1})$.

Proof. When $0 \leq b_1, \dots, b_{k-1} \leq m$ and $b_k \geq m+1$, the integral in $G(\beta; b)(s)$ has a pole of order $k-1$ at $s = -m$ arising from integration around the vertex $(u_1, \dots, u_{k-1}, u_k) = (0, \dots, 0, 1)$. Also the integrations around other vertices result in poles with order less than $k-1$. By the regularity of the

integral [3]

$$\begin{aligned}
 G(\beta; b)(-m) &= \text{the coefficient of } u_1^{m-b_1} \cdots u_{k-1}^{m-b_{k-1}} \text{ of the function} \\
 &\quad [A_1, U]^{-\beta_1-1} \cdots [A_q, U]^{-\beta_q-1} (1 - u_1 - \cdots - u_{k-1})^{b_k-m-1} \\
 &\quad \text{expanded at } u_1 = \cdots = u_{k-1} = 0 \\
 &= \frac{1}{(\mathbf{m} - \mathbf{b})!} D^{\mathbf{m}-\mathbf{b}} \{ [A_1, U]^{-\beta_1-1} \cdots [A_q, U]^{-\beta_q-1} \\
 &\quad \times (1 - u_1 - \cdots - u_{k-1})^{b_k-m-1} \}_{u_1=\cdots=u_{k-1}=0}.
 \end{aligned}$$

By Leibniz's rule for partial differential operators, we have

$$\begin{aligned}
 &\frac{1}{(\mathbf{m} - \mathbf{b})!} D^{\mathbf{m}-\mathbf{b}} [[A_1, U]^{-\beta_1-1} \cdots [A_q, U]^{-\beta_q-1} (1 - u_1 - \cdots - u_{k-1})^{b_k-m-1}] \\
 &= \sum_{\alpha_1 + \cdots + \alpha_{q+1} = \mathbf{m} - \mathbf{b}} \frac{1}{\alpha_1! \cdots \alpha_{q+1}!} \left[\prod_{i=1}^q D^{\alpha_i} [A_i, U]^{-\beta_i-1} \right] \\
 &\quad \times D^{\alpha_{q+1}} (1 - u_1 - \cdots - u_{k-1})^{b_k-m-1}.
 \end{aligned}$$

Replace $[A_j, U]$ by $a_{1,j}u_1 + \cdots + a_{k-1,j}u_{k-1} + a_{k,j}(1 - u_1 - \cdots - u_{k-1})$. Then an elementary calculation yields our values for $G(\beta; b)(-m)$.

The following lemma is needed in our proof of the next proposition.

Lemma. Let f be a continuous function on $[0, \infty)$, b_1, \dots, b_q nonnegative integers, and $E(h)$ the simplex defined by

$$0 \leq u_1 + \cdots + u_q \leq h, \quad u_1, \dots, u_q \geq 0, \quad h > 0.$$

Then

$$\begin{aligned}
 &\int_{E(h)} (u_1^{b_1} \cdots u_q^{b_q}) f(u_1 + \cdots + u_q) du_1 \cdots du_q \\
 &= \frac{b_1! \cdots b_q!}{(b_1 + \cdots + b_q + q - 1)!} \int_0^h f(t) t^{b_1 + \cdots + b_q + q - 1} dt.
 \end{aligned}$$

Proof. The lemma follows from a change of variables and the definition of a β -function of several variables.

Proposition 6. Let $R_j(X) = \sum_{i=q+1}^r a_{ji} X_i + \delta_j$ ($j = 1, \dots, k$). Then

$$\begin{aligned}
 &\int_{\Delta_k(X, \dots, X_q)} X^\beta P^m(X) dX_1 \cdots dX_q \\
 &= (m!)^k \sum_b \prod_{j=1}^k \frac{R_j^{b_j}(X)}{b_j!} \sum_{\alpha_1 + \cdots + \alpha_{q+1} = \mathbf{m} - \mathbf{b}} \prod_{j=1}^{k-1} \prod_{i=1}^q \left(\frac{a_{ki} - a_{ji}}{a_{ki}} \right)^{\alpha_{ji}} \\
 &\quad \times \prod_{i=1}^q \frac{(\beta_i + |\alpha_i|)!}{\beta_i! a_{ki}^{\beta_i+1}} (-1)^{|\alpha_{q+1}| + |\beta| + q} (b_k - m - 1)!.
 \end{aligned}$$

Here \mathbf{b} ranges over all k -tuple nonnegative integers (b_1, \dots, b_k) satisfying $0 \leq b_1, \dots, b_{k-1} \leq m$, $|\mathbf{b}| = km + |\beta| + 1$, and $b_k \geq m + 1$. Also $\alpha_j = (\alpha_{1j}, \dots, \alpha_{k-1,j})$ ($j = 1, \dots, q + 1$).

Proof. Rewrite the integral as

$$\begin{aligned} & \sum_{0 \leq b_1, \dots, b_{k-1} \leq m} \prod_{i=q+1}^r X_i^{\beta_i} \prod_{j=1}^{k-1} \frac{m!}{(m-b_j)! b_j!} R_j^{b_j}(X) \\ & \times \int_{\Delta_k(X, \dots, X_q)} \prod_{i=1}^q X_i^{\beta_i} \prod_{j=1}^{k-1} (a_{j1} X_1 + \dots + a_{jq} X_q)^{m-b_j} L_k(X) dX_1 \dots dX_q. \end{aligned}$$

Now let $X_i = -u_i R_k(X)/a_{ki}$ ($i = 1, \dots, q$). Then $\Delta_k(X_1, \dots, X_q)$ is transformed into the standard q -simplex E^q defined by

$$E^q : u_1 \geq 0, \dots, u_q \geq 0, \quad 1 - u_1 - \dots - u_q \geq 0.$$

Consequently, we have

$$\begin{aligned} & \int_{\Delta_k(X, \dots, X_q)} \prod_{i=1}^q X_i^{\beta_i} \prod_{j=1}^{k-1} (a_{j1} X_1 + \dots + a_{jq} X_q)^{m-b_j} L_k(X) dX_1 \dots dX_q \\ & = \prod_{i=1}^q \left(-\frac{1}{a_{ki}} \right)^{\beta_i+1} [R_k(X)]^{b_k} \int_{E^q} \prod_{i=1}^q u_i^{\beta_i} \prod_{j=1}^{k-1} \left(-\frac{a_{j1} u_1}{a_{k1}} - \dots - \frac{a_{jq} u_q}{a_{kq}} \right)^{m-b_j} \\ & \quad \times (1 - u_1 - \dots - u_q)^m du_1 \dots du_q. \end{aligned}$$

Note that

$$\begin{aligned} & \left(-\frac{a_{j1}}{a_{k1}} u_1 - \dots - \frac{a_{jq}}{a_{kq}} u_q \right)^{m-b_j} \\ & = \left[\left(\frac{a_{k1} - a_{j1}}{a_{k1}} u_1 \right) + \dots + \left(\frac{a_{kq} - a_{jq}}{a_{kq}} u_q \right) - (u_1 - \dots - u_q) \right]^{m-b_j} \\ & = \sum_{\alpha_{j1} + \dots + \alpha_{j,q+1} = m-b_j} \frac{(m-b_j)!}{\alpha_{j1}! \dots \alpha_{j,q+1}!} \left(\frac{a_{k1} - a_{j1}}{a_{k1}} u_1 \right)^{\alpha_{j1}} \dots \left(\frac{a_{kq} - a_{jq}}{a_{kq}} u_q \right)^{\alpha_{jq}} \\ & \quad \times (-1)^{\alpha_{j,q+1}} (u_1 + \dots + u_q)^{\alpha_{j,q+1}}. \end{aligned}$$

Hence we have

$$\begin{aligned} & \int_{E^q} \prod_{i=1}^q u_i^{\beta_i} \prod_{j=1}^{k-1} \left(-\frac{a_{j1} u_1}{a_{k1}} - \dots - \frac{a_{jq} u_q}{a_{kq}} \right)^{m-b_j} (1 - u_1 - \dots - u_q)^m du_1 \dots du_q \\ & = \sum_{\alpha_1 + \dots + \alpha_{q+1} = m-b} \left(\prod_{i=1}^q \prod_{j=1}^{k-1} \left(\frac{a_{ki} - a_{ji}}{a_{ki}} \right)^{\alpha_{ji}} \cdot \frac{1}{a_{ji}!} \right) (-1)^{|\alpha_{q+1}|} \\ & \quad \times \int_{E^q} u_1^{\beta_1 + |\alpha_1|} \dots u_q^{\beta_q + |\alpha_q|} (1 - u_1 - \dots - u_q)^m \\ & \quad \times (u_1 + \dots + u_q)^{|\alpha_{q+1}|} du_1 \dots du_q. \end{aligned}$$

By our lemma, the integral is equal to

$$\frac{[\prod_{i=1}^q (\beta_i + |\alpha_i|)!] m! (b_k - m - 1)!}{b_k!}.$$

Combining the previous propositions, we get the following proposition, which is precisely what we needed.

Proposition 7. *With notation as above, we have*

$$\begin{aligned}
 & (-1)^m m! \sum_{\alpha} \beta_1! \cdots \beta_q! (-1)^{\beta_{q+1} + \cdots + \beta_r} \left[\prod_{i=q+1}^r \frac{B_{\alpha_i} + \beta_i + 1}{\alpha_i + \beta_i + 1} \right] \frac{(-1)^{\alpha_{r+1}}}{\alpha_{r+1}!} \\
 & \quad \times \sum_b \prod_{j=1}^q \prod_{i=q+1}^{r+1} \frac{1}{\alpha_{ji}!} G(\beta; b)(-m) \\
 & = J^{r-q} \left[\prod_{j=1}^k \int_{\Delta_k(X, \dots, X_q)} X^{\beta} P^m(X) dX_1 \cdots dX_q \right].
 \end{aligned}$$

Here α ranges over all nonnegative integers $(\alpha_{q+1}, \dots, \alpha_r, \alpha_{r+1})$ such that $|\alpha| = km + |\beta| + q$ in the first summation, and the relation between b and α is

$$b = (b_1, \dots, b_k) = \left(\sum_{i=q+1}^r \alpha_{1i}, \dots, \sum_{i=q+1}^r \alpha_{ki} \right)$$

if $\alpha_i = \alpha_{1i} + \cdots + \alpha_{ki}$ ($i = q+1, \dots, r+1$).

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