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"COMPLETE-SIMPLE" DISTRIBUTIVE LATTICES

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ABSTRACT. It is well known that the only simple distributive lattice is the two-element chain. We can generalize the concept of a simple lattice to complete lattices as follows: a complete lattice is *complete-simple* if it has only the two trivial complete congruences. In this paper we show the existence of infinite complete-simple distributive lattices.

1. INTRODUCTION

A number of authors (Freese, Grätzer, Johnson, Lakser, Reuter, Schmidt, Teo, Wille, and Wolk, see the references) have investigated the lattice of all complete congruence relations of a complete lattice. They have proved representation theorems of the form: given a complete lattice L, a complete lattice K is constructed such that the lattice of all complete congruence relations of K is isomorphic to L. This result was first proved by Grätzer in [3]; a planar K was constructed by Grätzer and Lakser in [6], and a modular K was constructed by Freese, Grätzer, and Schmidt in [1]. All these constructions were based on manipulating prime intervals in various ways.

It was observed in [1] that such techniques cannot be applied to complete distributive lattices since the congruence relation generated by a prime interval is always a complete congruence relation. Let us call a complete lattice *complete-simple* if it has only the two trivial complete congruences. It follows from the observation quoted above that a complete-simple distributive lattice containing a prime interval is the two-element chain.

The question naturally arises whether there is a complete-simple distributive lattice without a prime interval.

Theorem. *There exists a complete-simple distributive lattice K with more than two elements.*

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2. Notation

For the notation and basic concepts, we refer the reader to [2].

Let L be a complete lattice. A *complete congruence relation* Θ of L is a congruence relation for which the Substitution Property holds for arbitrary joins and meets, that is, if $x_i \equiv y_i\ (\Theta)$ for $i \in I$, then $\bigvee(x_i \mid i \in I) \equiv \bigvee(y_i \mid i \in I)$ (Θ) and $\bigwedge(x_i \mid i \in I) \equiv \bigwedge(y_i \mid i \in I)$ (Θ). The smallest and largest congruence, denoted by ω and ι, respectively, are complete. The complete congruences of L form a complete lattice; if this lattice contains only ω and ι, then we shall call L *complete-simple.*

$\mathbb{Q}$ and $\mathbb{R}$ will denote the chain of rational and real numbers, respectively.

Let L_i, $i \in I$, be lattices. Then $\prod(L_i \mid i \in I)$ denotes their direct product. If $\mathbf{t} \in \prod(L_i \mid i \in I)$, then $\mathbf{t}(i) \in L_i$ is the ith component of $\mathbf{t}$.

3. A unary operation

The construction is based on a special unary operation x^+ on $\mathbb{Q}$. This will be introduced in this section.

For a unary operation x^+ on a set A, we will use the following notation for iterated applications: $x^{[0]} = x$, $x^{[1]} = x^+, \dots, x^{[n+1]} = (x^{[n]})^+, \dots$. For $a \in A$, set $a^{\mathrm{v}} = \langle a^{[0]}, a^{[1]}, \dots \rangle$ and $H_a = \{\, b \mid b \in A \text{ and } b^+ = a \,\}$.

Lemma 1. *For every infinite set A, there exists a unary operation x^+ satisfying the following two properties*:

(1) *$a^{[n]} \neq a^{[m]}$ for all $a \in A$ and for all natural numbers $n \neq m$.*
(2) *There exists a bijection between H_a and A for every $a \in A$.*

Proof. Without loss of generality, we can assume that $A = B^2 \times \mathbb{Z}$, where $|A| = |B|$ and $\mathbb{Z}$ is the set of integers. Since $|B|^2 = |B|$, there is a bijection φ between B and B^2. For $a = \langle \langle b_1, b_2 \rangle, i \rangle$, define $a^+ = \langle \varphi(b_1), i+1 \rangle$.

The $\mathbb{Z}$-coordinate of $a^{[k]}$ is $i+k$; therefore, $a^{[n]} = a^{[m]}$ implies that $i+n = i+m$, so $n = m$, verifying (1).

Since φ is onto, we can choose a $b \in B$ with $\varphi(b) = \langle b_1, b_2 \rangle$. Then $H_a = \{\, \langle \langle b, x \rangle, i-1 \rangle \mid x \in B \,\}$, implying (2). □

Henceforth, let $A = \mathbb{Q}$, and let x^+ be a unary operation on $\mathbb{Q}$ satisfying (1) and (2) of Lemma 1. We take a family of pairwise disjoint chains $(\mathbb{R}_i \mid i \in \mathbb{Q})$, where each $\mathbb{R}_i$ is isomorphic to the chain $\mathbb{R}$. We denote by $\mathbb{Q}_i$ the chain of rational numbers in $\mathbb{R}_i$, $i \in \mathbb{Q}$. By the second condition of Lemma 1, there is a bijection between H_i and $\mathbb{Q}$ for every $i \in \mathbb{Q}$; hence, there is a bijection α_i, $i \in \mathbb{Q}$, between H_i and $\mathbb{Q}_i$. We keep the α_i, $i \in \mathbb{Q}$, fixed for the rest of this paper.

Let ϱ denote the disjoint union of the bijections α_i, $i \in \mathbb{Q}$. More formally, we define a map ϱ of $\mathbb{Q}$ into $\bigcup(\mathbb{Q}_i \mid i \in \mathbb{Q})$ as follows:

Let $i \in \mathbb{Q}$. Then $\varrho(i) = \alpha_{i^+}(i) \in \mathbb{Q}_{i^+} \subseteq R_{i^+}$.

Observe that $\varrho(i)$ is an element of $\mathbb{Q}_{i^+}$ and, in fact, every element of $\mathbb{Q}_{i^+}$ is of the form $\varrho(i)$ for some $i \in \mathbb{Q}$. Obviously, ϱ is a bijection between $\mathbb{Q}$ and $\bigcup(\mathbb{Q}_i \mid i \in \mathbb{Q})$.

4. Construction

Let $P = \prod(\mathbb{R}_i \mid i \in \mathbb{Q})$. We adjoin to P a zero, $\mathbf{0}$, and a unit element, $\mathbf{1}$, to obtain the poset $\overline{P}$. Let $\mathbf{t} \in P$; for $i \in \mathbb{Q}$, we introduce Condition (C_i) for $\mathbf{t}$ as follows:

(C_i) holds for $\mathbf{t}$ if and only if $\mathbf{t}(i) = 0$; or $\mathbf{t}(i) > 0$ and $\mathbf{t}(i^+) > \varrho(i)$; or $\mathbf{t}(i) < 0$ and $\mathbf{t}(i^+) < \varrho(i)$.

Condition (C) holds for $\mathbf{t}$ if (C_i) holds for $\mathbf{t}$ for all $i \in \mathbb{Q}$.

Now we define the subset S of P as the set of all $\mathbf{t} \in P$ for which Condition (C) holds. Define $\overline{S} = S \cup \{\mathbf{0}, \mathbf{1}\}$.

Lemma 2. *S is a sublattice of P; hence, S is a distributive lattice. $\overline{S}$ is a complete distributive lattice.*

Proof. Observe that $S \neq \varnothing$. Indeed, let $\mathbf{z} \in P$ be defined by $\mathbf{z}(i) = 0$ for all $i \in \mathbb{Q}$. Condition (C) obviously holds for $\mathbf{z}$, hence, $\mathbf{z} \in S$.

Let $\mathbf{s}, \mathbf{t} \in S$; we form $\mathbf{s} \vee \mathbf{t}$ in P. We claim that $\mathbf{s} \vee \mathbf{t} \in S$, that is, (C_i) holds for $\mathbf{s} \vee \mathbf{t}$, for $i \in \mathbb{Q}$. Indeed, if $(\mathbf{s} \vee \mathbf{t})(i) > 0$, then $\mathbf{s}(i) \vee \mathbf{t}(i) = (\mathbf{s} \vee \mathbf{t})(i) > 0$; therefore, $\mathbf{s}(i) > 0$ or $\mathbf{t}(i) > 0$ (since $\mathbb{R}_i$ is a chain). Since (C_i) holds for $\mathbf{s}$ and $\mathbf{t}$, $\mathbf{s}(i^+) > \varrho(i)$ or $\mathbf{t}(i^+) > \varrho(i)$, concluding that $(\mathbf{s} \vee \mathbf{t})(i^+) > \varrho(i)$. If $(\mathbf{s} \vee \mathbf{t})(i) < 0$, then $\mathbf{s}(i) \vee \mathbf{t}(i) = (\mathbf{s} \vee \mathbf{t})(i) < 0$; therefore, $\mathbf{s}(i) < 0$ and $\mathbf{t}(i) < 0$. It follows that $\mathbf{s}(i^+) < \varrho(i)$ and $\mathbf{t}(i^+) < \varrho(i)$; therefore, $(\mathbf{s} \vee \mathbf{t})(i^+) = \mathbf{s}(i^+) \vee \mathbf{t}(i^+) < \varrho(i)$ (since $\mathbb{R}_i$ is a chain). Similarly, we prove that $\mathbf{s} \wedge \mathbf{t} \in S$. Hence, S is a lattice; in fact, since it is a sublattice of P, it is a distributive lattice.

Next we prove that $\overline{S}$ is a complete lattice. In order to do this, first, for every $\mathbf{v} \in \overline{P}$, we define $\overline{\mathbf{v}} \in \overline{P}$ as follows: If $\mathbf{v} \in \overline{P} - P$, then $\overline{\mathbf{v}} = \mathbf{v}$. If $\mathbf{v} \in P$, then let $\mathbf{v}_0 = \mathbf{v}$. Consider the set $U_{\mathbf{v}_0} = U_0$ of all $i \in \mathbb{Q}$ such that $\mathbf{v}_0(i) < 0$ and $\mathbf{v}_0(i^+) \nless \varrho(i)$. Define $\mathbf{v}_1 \in P$: $\mathbf{v}_1(i) = 0$ for $i \in U_0$ and $\mathbf{v}_1(j) = \mathbf{v}_0(j)$ for $j \in \mathbb{Q} - U_0$. Inductively, let n be a natural number > 1, and let $\mathbf{v}_n$ be defined. Then let $U_n = U_{\mathbf{v}_n}$ and $\mathbf{v}_{n+1} = (\mathbf{v}_n)_1$. Finally, let $\overline{\mathbf{v}} = \bigvee(\mathbf{v}_n \mid n < \omega)$.

Observe that $\mathbf{v} \leq \overline{\mathbf{v}}$ and if $\overline{\mathbf{v}}(i) \neq \mathbf{v}(i)$ for some $i \in \mathbb{Q}$, then $\overline{\mathbf{v}}(i) = 0$.

Let T be a subset of S, and let $\mathbf{u} = \bigvee T$ in $\overline{P}$. If $\mathbf{u} \in \overline{S}$, then $\mathbf{u} = \bigvee T$ in $\overline{S}$. So we can assume that $\mathbf{u} \in \overline{P} - \overline{S}$. We shall prove that, in this case, $\overline{\mathbf{u}} = \bigvee T$ in $\overline{S}$.

To verify that $\overline{\mathbf{u}} \in \overline{S}$, it is sufficient to prove that if $\overline{\mathbf{u}} \notin \{\mathbf{0}, \mathbf{1}\}$, then $\overline{\mathbf{u}} \in S$. Let $i \in \mathbb{Q}$, $\overline{\mathbf{u}}(i) < 0$, and $\overline{\mathbf{u}}(i^+) \nless \varrho(i)$. Then choose the natural numbers k and m so that $\overline{\mathbf{u}}(i) = \mathbf{u}_k(i)$ and $\overline{\mathbf{u}}(i^+) = \mathbf{u}_m(i^+)$. With $n = \max\{k, m\}$, we have $\mathbf{u}_n(i) < 0$ and $\mathbf{u}_n(i^+) \nless \varrho(i)$. Therefore, $i \in U_{\mathbf{u}_n}$, hence $\mathbf{u}_n(i)$ had to be corrected at step $n+1$. We conclude that $\overline{\mathbf{u}}(i) = \mathbf{u}_{n+1}(i) = 0$, contradicting the assumption that $\overline{\mathbf{u}}(i) < 0$. Next let $i \in \mathbb{Q}$, $\overline{\mathbf{u}}(i) > 0$, and $\overline{\mathbf{u}}(i^+) \ngtr \varrho(i)$. Since $\overline{\mathbf{u}}(i) > 0$, it follows that $\overline{\mathbf{u}}(i) = \mathbf{u}(i)$. Therefore, $\overline{\mathbf{u}}(i) = \mathbf{u}(i) = \bigvee(\mathbf{v}(i) \mid \mathbf{v} \in T) > 0$, hence $\mathbf{v}(i) > 0$ for some $\mathbf{v} \in T$ (since $\mathbb{R}_i$ is a chain). Since $\mathbf{v} \in T$, (C_i) holds for $\mathbf{v}$, so $\varrho(i) < \mathbf{v}(i^+) \leq \mathbf{u}(i^+) \leq \overline{\mathbf{u}}(i^+)$, contradicting the assumption that $\overline{\mathbf{u}}(i^+) \ngtr \varrho(i)$. We conclude that Condition (C_i) holds for $\overline{\mathbf{u}}$.

Finally, we have to verify that $\overline{\mathbf{u}}$ is the complete join of T in $\overline{S}$. Let $\mathbf{s} \in \overline{S}$ be an upper bound of T, that is, $\mathbf{t} \leq \mathbf{s}$, for all $\mathbf{t} \in T$. We want to show that $\overline{\mathbf{u}} \leq \mathbf{s}$. This is obvious if $\mathbf{s} \in \{\mathbf{0}, \mathbf{1}\}$. So let $\mathbf{s} \notin \{\mathbf{0}, \mathbf{1}\}$, that is, let $\mathbf{s} \in S$. Obviously, $\mathbf{u} \leq \mathbf{s}$. Now we prove that $\mathbf{u}_n \leq \mathbf{s}$ by induction on n. For $n = 0$, we have $\mathbf{u}_0 = \mathbf{u}$, and we already know that $\mathbf{u} \leq \mathbf{s}$. Let $\mathbf{u}_n \leq \mathbf{s}$; for every $i \in \mathbb{Q}$, we have to verify that $\mathbf{u}_{n+1}(i) \le \mathbf{s}(i)$. If $i \notin U_n$, then $\mathbf{u}_n(i) = \mathbf{u}_{n+1}(i)$, so $\mathbf{u}_{n+1}(i) \le \mathbf{s}(i)$ follows from the induction hypothesis. If $i \in U_n$, then $\mathbf{u}_n(i) < 0$ and $\mathbf{u}_n(i^+) \not< \varrho(i)$. It follows that $\mathbf{s}(i^+) \ge \mathbf{u}_n(i^+) \ge \varrho(i)$. Since $\mathbf{s}$ satisfies Condition (C), we conclude that $\mathbf{s}(i) \ge 0$. Therefore, $\mathbf{s}(i) \ge 0 = \mathbf{u}_{n+1}(i)$, completing the induction. $\square$

5. Complete congruences

In this section, we shall prove that $\overline{S}$ is a complete-simple lattice. Let Θ be a complete congruence relation of $\overline{S}$, and let $\Theta > \omega$. We are going to prove that $\mathbf{0} \equiv \mathbf{1}$ (Θ), that is, $\Theta = \iota$.

We introduce some notation. With $\mathfrak{a} = \langle a_0, a_1, \dots \rangle \in \mathbb{R}^\omega$ and $k \in \mathbb{Q}$, we associate the element $\mathfrak{a}^k$ of P defined by $\mathfrak{a}^k(k^{[n]}) = a_n$, for $n = 0, 1, \dots$, and $\mathfrak{a}^k(l) = 0$, otherwise. It is obvious that $\mathfrak{a}^k \in \overline{S}$ iff Condition (C_l) holds for all $l = k^{[n]}$, $n = 0, 1, \dots$.

Since $\Theta > \omega$, there exist $\mathbf{s}$, $\mathbf{t} \in \overline{S}$ such that $\mathbf{s} < \mathbf{t}$ and $\mathbf{s} \equiv \mathbf{t}$ (Θ). Without loss of generality, we can assume that $\mathbf{s}$, $\mathbf{t} \in S$.

There exists a $j \in \mathbb{Q}$ such that $\mathbf{s}(j) < \mathbf{t}(j)$. Recall that α_j is a bijection between H_j and $\mathbb{Q}_j$. Since $\mathbb{Q}_j$ is dense in $\mathbb{R}_j$, there is an $i \in H_j$ such that

$$\mathbf{s}(j) = \mathbf{s}(i^+) < \varrho(i) < \mathbf{t}(i^+) = \mathbf{t}(j).$$

(By definition, $j = i^+$ and $\varrho(i) = \alpha_{i^+}(i)$.)

Henceforth, $\mathbf{s}$, $\mathbf{t}$, and i will refer to these elements. Define $s_n = \mathbf{s}(i^{[n]})$ and $t_n = \mathbf{t}(i^{[n]})$, for $n = 0, 1, \dots$. Let $\varrho(n)$ stand for $\varrho(i^{[n]})$, $n = 0, 1, \dots$. Then the condition on $\mathbf{s}$, $\mathbf{t}$, and i can be rewritten as follows:

$$s_1 < \varrho(0) < t_1.$$

For $\mathfrak{a} = \langle a_0, a_1, \dots \rangle \in \mathbb{R}^\omega$ and for this $i \in \mathbb{Q}$, we identify $\mathfrak{a}$ with $\mathfrak{a}^i$; in other words, $\mathfrak{a} = \langle a_0, a_1, \dots \rangle$ will be regarded as an element of P. Then $\langle s_0, s_1, s_2, \dots \rangle$, $\langle t_0, t_1, t_2, \dots \rangle \in S$ and $\langle 0, s_1, s_2, \dots \rangle$, $\langle 0, t_1, t_2, \dots \rangle \in S$.

From $\mathbf{s} \equiv \mathbf{t}$ (Θ), we conclude that

$$\begin{aligned}(\mathbf{s} \wedge \langle 0, t_1, t_2, \dots \rangle) &\vee \langle 0, s_1, s_2, \dots \rangle \\ &\equiv (\mathbf{t} \wedge \langle 0, t_1, t_2, \dots \rangle) \vee \langle 0, s_1, s_2, \dots \rangle \ (\Theta),\end{aligned}$$

that is,

$$\langle 0, s_1, s_2, \dots \rangle \equiv \langle 0, t_1, t_2, \dots \rangle \quad (\Theta).$$

Let us choose u, $v \in \mathbb{R}$ so that $0 < u$ and $\varrho(0) < v \le t_1$; if $\varrho(0) < 0$, then we further assume that $v < 0$. Define the element $\langle u, v, v_2, v_3, \dots \rangle$ of S as follows: if $0 \le \varrho(0)$, let $v_i = t_i$, $i = 2, 3, \dots$; if $\varrho(0) < 0$, then $v_i = s_i$, $i = 2, 3, \dots$. It is clear that $\langle u, v, v_2, v_3, \dots \rangle \in S$ and that v_i, $i = 2, 3, \dots$, do not depend on the choice of u and v. Also, $\langle u, t_1, t_2, \dots \rangle \in S$.

Computing with these elements, we conclude from the last congruence that

$$\begin{aligned}(\langle 0, s_1, s_2, s_3, \dots \rangle &\vee \langle u, v, v_2, v_3, \dots \rangle) \wedge \langle u, t_1, t_2, t_3, \dots \rangle \\ &\equiv (\langle 0, t_1, t_2, t_3, \dots \rangle \vee \langle u, v, v_2, v_3, \dots \rangle) \wedge \langle u, t_1, t_2, t_3, \dots \rangle \ (\Theta),\end{aligned}$$

that is,

$$\langle u, v, v_2, v_3, \dots\rangle \equiv \langle u, t_1, t_2, t_3, \dots\rangle \quad (\Theta).$$

Observe that $s_n \le v_n \le t_n$ holds for all $n > 1$.

For any $0 < u$, for all $v \in \mathbb{R}$ satisfying $\varrho(0) < v \le t_1$ and $v < 0$ if $\varrho(0) < 0$, we obtain a congruence as above. Forming the complete meet of these congruences, we get the congruence

$$\bigwedge(\langle u, v, v_2, v_3, \dots\rangle \mid \varrho(0) < v \le t_1 \text{ and } v < 0 \text{ if } \varrho(0) < 0)$$
$$\equiv \langle u, t_1, t_2, \dots\rangle \ (\Theta).$$

Since $\bigwedge(v \mid \varrho(0) < v \le t_1 \text{ and } v < 0 \text{ if } \varrho(0) < 0) = \varrho(0)$, the complete meet $\bigwedge(\langle u, v, \dots\rangle \mid \varrho(0) < v \le t_1 \text{ and } v < 0 \text{ if } \varrho(0) < 0)$ in $\overline{S}$ is $\langle 0, \varrho(0), v_2, v_3, \dots\rangle$, so we have arrived at the congruence

$$\langle 0, \varrho(0), v_2, v_3, \dots\rangle \equiv \langle u, t_1, t_2, t_3, \dots\rangle \quad (\Theta),$$

where $s_n \le v_n \le t_n$ for all $n > 1$. For every $u > 0$, we obtain a congruence as above. We now form the complete join of these congruences:

$$\bigvee(\langle 0, \varrho(0), v_2, v_3, \dots\rangle \mid u > 0) \equiv \bigvee(\langle u, t_1, t_2, t_3, \dots\rangle \mid u > 0) \quad (\Theta).$$

This is obviously of the form

$$\mathfrak{y} = \langle 0, \varrho(0), v_2, v_3, \dots\rangle \equiv \mathbf{1} \quad (\Theta),$$

where $s_n \le v_n \le t_n$ for all $n > 1$. It follows that

$$\langle 0, s_1, s_2, \dots\rangle \le \mathfrak{y} \le \langle 0, t_1, t_2, \dots\rangle.$$

We can proceed similarly for $u < 0$ and $s_1 \le v < \varrho(0)$ (where $v > 0$ if $\varrho(0) > 0$), forming the complete join of the congruences for fixed u and all v, and then forming the complete meet for all u. We obtain a congruence of the form

$$\mathfrak{z} = \langle 0, \varrho(0), v_2', v_3', \dots\rangle \equiv \mathbf{0} \quad (\Theta),$$

where $s_n \le v_n' \le t_n$ for all $n > 1$; therefore,

$$\langle 0, s_1, s_2, \dots\rangle \le \mathfrak{z} \le \langle 0, t_1, t_2 \dots\rangle.$$

Now the last two congruences along with the congruence $\langle 0, s_1, s_2, \dots\rangle \equiv \langle 0, t_1, t_2, \dots\rangle$ (Θ), yield $\mathbf{0} \equiv \mathbf{1}$ (Θ), completing the proof of $\Theta = \iota$.

This completes the proof of the theorem.

6. Concluding comments

Let $\mathfrak{c}$ denote the power of the continuum. Obviously, the complete-simple lattice K constructed for the theorem is of power $\mathfrak{c}$. This is the smallest possible cardinality. Indeed, let K be a complete-simple distributive lattice. Let C be a maximal chain in K. By the observation quoted from [1] in the introduction, if C contains a prime interval, then C is the two-element chain and so is K. Therefore, if K has more than two elements, then C is dense. It follows that C contains a subchain isomorphic to $\mathbb{Q}$. Since C is complete, it contains a subchain isomorphic to $\mathbb{R}$. We conclude that $\mathfrak{c} \le |C| \le |K|$.

We can construct a complete-simple distributive lattice of any cardinality $\mathfrak{n} \geq \mathfrak{c}$. The proof of the theorem gives us such a K for $\mathfrak{n} = \mathfrak{c}$. Now let α be any ordinal of cardinality $\mathfrak{n} > \mathfrak{c}$. Utilizing the fact that $\mathbf{0}$ is meet-irreducible and $\mathbf{1}$ is join-irreducible in the lattice K constructed for the theorem, we construct a complete-simple distributive lattice K_γ, for every $\gamma \leq \alpha$, in which the zero $\mathbf{0}_\gamma$ is meet-irreducible and the unit $\mathbf{1}_\gamma$ is join-irreducible as follows:

$K_0 = K$; if $\gamma = \beta + 1$, then

$$K_\gamma = (K_\beta - \{\mathbf{0}_\beta, \mathbf{1}_\beta\})^2 \cup \{\mathbf{0}_\gamma, \mathbf{1}_\gamma\};$$

and K_β has a natural embedding into $K_{\beta+1}$; if γ is a limit ordinal, then K_γ is the direct limit of the K_δ, $\delta < \gamma$.

It is easy to check that K_α is a complete-simple distributive lattice of cardinality $\mathfrak{n}$.

An alternative proof, by modifying the construction for the theorem, was carried out by Johnson.

The topic described in the introduction, namely, the representation of a complete lattice as the lattice of all complete congruence relations of a complete lattice, was extended to the $\mathfrak{m}$-complete case by Grätzer and Lakser [7] and Grätzer and Schmidt [9], where $\mathfrak{m}$ is an infinite regular cardinal satisfying $\mathfrak{m} > \aleph_0$.

Let $\mathfrak{m}$ be an infinite regular cardinal. A lattice K is $\mathfrak{m}$-*complete* if $\bigvee X$ and $\bigwedge X$ exist in K whenever $X \subseteq K$ and $0 < |X| < \mathfrak{m}$. A congruence relation Θ of an $\mathfrak{m}$-complete lattice K is an $\mathfrak{m}$-*complete congruence relation* if the *Substitution Property* holds for fewer than $\mathfrak{m}$ elements, that is, if $x_i \equiv y_i$ (Θ) for $i \in I$, and $0 < |I| < \mathfrak{m}$, then

$$\bigvee(x_i \mid i \in I) \equiv \bigvee(y_i \mid i \in I) \quad (\Theta),$$

and dually. An $\mathfrak{m}$-complete lattice K is called $\mathfrak{m}$-*simple* if it has only the two trivial $\mathfrak{m}$-complete congruence relations.

Now we state the analogue of the theorem for the $\mathfrak{m}$-complete case; in this version of the theorem, we also state some properties of the lattice K we construct.

Theorem$'$. *Let* $\mathfrak{m} > \aleph_0$ *be an infinite regular cardinal. There exists an* $\mathfrak{m}$-*simple distributive lattice* K *with more than two elements. The lattice* K *can be constructed to have the following additional properties*:

(1) K *is complete.*
(2) K *is self-dual.*
(3) *The zero of* K *is meet-irreducible and it is* $\mathfrak{m}$-*complete meet-reducible.*
(4) *The unit of* K *is join-irreducible and it is* $\mathfrak{m}$-*complete join-reducible.*

The lattice K constructed for the theorem satisfies (1); it is clear that properties (3) and (4) hold for K. Since $\mathfrak{m} > \aleph_0$ and $\mathfrak{c} \geq \aleph_1$, it is enough to show that K is $\aleph_1$-simple.

Whenever we form a join $\bigvee X = a$ in some $\mathbb{R}_i$, we can take a countable subset X_1 of X with $\bigvee X_1 = a$, and dually. So in the proof that K is complete-simple (see §5), all the applications of the complete Substitution Property could be replaced by applications of the $\aleph_1$-Substitution Property. Therefore, K is $\aleph_1$-simple.

Finally, one can argue that by carefully choosing the function ϱ, the lattice K is self-dual. It is much simpler, however, to verify (2) by observing that

$$((K - \{\mathbf{0}_K, \mathbf{1}_K\}) \times (\widetilde{K} - \{\mathbf{0}_K, \mathbf{1}_K\})) \cup \{\mathbf{0}, \mathbf{1}\},$$

where $\widetilde{K}$ is the dual of K, is self-dual, and satisfies all the requirements of Theorem$'$.

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