# ADDING DOMINATING REALS WITH THE RANDOM ALGEBRA 

HAIM JUDAH AND SAHARON SHELAH<br>(Communicated by Andreas R. Blass)


#### Abstract

We show that there are two models $M \subseteq N$ such that by forcing with (Random) ${ }^{M}$ over $N$ we add dominating reals. This answers a question of A. Miller.


Let $R$ be the random real forcing. It is well known that $R$ is an $\omega^{\omega}$-bounding forcing notion, that is

$$
\left(\forall f \in \omega^{\omega} \cap V^{R} \exists g \in \omega^{\omega} \cap V\right)(\forall n \in \omega)(f(n)<g(n)) .
$$

For more detail and notation the reader should see [ Ku ]. The following is also known [BJ2]:

$$
\text { In } V^{R \times R} \text { there are Cohen reals over } V \text {. }
$$

From this we can conclude the following.
There are models $M \subset N$ such that in $N^{R \cap M}$ there are unbounded reals over $N$. (Take $N=M^{R}$ and use the previous result.)

After this it was natural to ask:
(1) Are there $M \subseteq N$ such that in $N^{R \cap M}$ there are dominating reals over $N$ ?

Let us introduce more notation. Let $I$ be an ideal of subsets of $\mathbf{R}$. Then we define $K_{A}(I)$ as the cardinality of the smallest family of elements of $I$ whose union is not in $I . K_{B}(I)$ is the cardinality of the smallest covering of the reals by elements of $I . b$ is the cardinality of the smallest family of functions from $\omega$ to $\omega$, which is unbounded. Miller [Mi] proved that

$$
\operatorname{cof}\left(K_{B}(\text { Meager })\right)>\omega .
$$

More generally, Bartoszynski and Judah [BJ1] proved that

$$
\operatorname{cof}\left(K_{B}(\text { Meager })\right) \geq K_{A}(\text { Measure zero }) .
$$

It is an open problem if

$$
\operatorname{cof}\left(K_{B}(\text { Meager })\right) \geq K_{A}(\text { Meager }) .
$$

[^0]After the Miller result it was natural to ask

$$
\text { Is } \operatorname{cof}\left(K_{B}(\text { Measure zero })\right)>\omega ?
$$

This question remains open and has produced a lot of development. The only positive result in this direction is the following theorem of Bartoszynski [Ba]:

$$
\operatorname{cof}\left(K_{B}(\text { measure zero })\right)>\omega \quad \text { if } b \geq K_{B} \text { (measure zero). }
$$

When we started working on this problem we proposed the iteration

$$
\bar{Q}=\left\langle P_{\alpha}, \mathbf{Q}_{\beta}: \alpha \geq \aleph_{\omega+1}, \beta<\aleph_{\omega+1}\right\rangle
$$

satisfying
(i) $\vDash$ " $\mathbf{Q}_{0}$ adds $\aleph_{\omega}$-many Cohen reals",
(ii) $\vDash_{P_{\beta}}$ " $\mathbf{Q}_{\beta}$ is a subalgebra of Random reals of cardinality less than $\aleph_{\omega} "$;
(iii) The sequence is generic enough in order to force with every possible subalgebra of Random reals.
Our conjecture was

$$
V^{P_{\aleph_{\omega+1}}} \Vdash^{\prime} \text { " } K_{B} \text { (measure zero) }=\aleph_{\omega} "
$$

After hard work we started thinking that maybe we were missing something and we asked:

$$
\text { Maybe } V^{P_{\aleph_{\omega+1}}} \Vdash b \geq \aleph_{\omega} \text { ? }
$$

It is easy to see (by [Ba]) that this is true if for some $\beta<\aleph_{\omega+1}$

$$
\Vdash_{P_{\beta}} " \mathbf{Q}_{\beta} \text { adds a dominating real". }
$$

Therefore the question was: Does there exist $R^{\prime} \subseteq R$ such that

$$
\Vdash_{R^{\prime}} \text { "add a dominating real"? }
$$

We show in [JS2] that under $C H$, or under $K_{B}$ (Meager) $=2^{\aleph_{0}}$, there exists such subalgebras of $R$. And using this and Bartoszynski's result, it is not hard to see that

$$
V^{P_{\aleph_{\omega+1}}} \vDash K_{B}(\text { measure zero })=\aleph_{\omega+1}
$$

The construction in [JS2] was not strong enough to solve Miller's question (1). That is, our example was not the random algebra restricted to some inner model; therefore, we thought that we could change condition (ii) of the iteration to
$\left(\right.$ ii)* $\Vdash_{P_{\beta}}$ " $\mathbf{Q}_{\beta}=M \cap R$ for some inner model $M \vDash 2^{\aleph_{0}}=\aleph_{n}$, for $n<\omega$ ".
Again we were unable to show that this new iteration gives the desired model, and we recalled Miller's question (1).

In this work we will answer Miller's question (1) positively by showing:
There are two models $M \subseteq N$ such that forcing over $N$ with $R \cap M$ we add dominating reals. We sketch the construction as follows: We start for simplicity with $V=L$. Then we add $\aleph_{2}$ Cohen reals. After this we add, with finite support iteration, a sequence $\left\langle r_{i}: i<\omega_{1}\right\rangle$ of positive sets by a forcing notion, which is Souslin (see [JS1]) and has the appearance of Amoeba forcing. Then we let $M=V\left[\left\langle r_{i}: i<\omega_{1}\right\rangle\right]$ and $N=V\left[\aleph_{2}\right.$ - Cohen $]\left[\left\langle r_{i}: i<\omega\right\rangle\right]$.

The notation is standard and the rest of the paper is devoted to building the models.

1. Assumption. Let $\bar{W}=\left\langle W_{n}: n<\omega\right\rangle$ be a sequence of pairwise disjoint subsets of $\omega$. We also assume
(0) $W_{n}$ is infinite.
(1) For every $n<\omega, m<\omega$, and $k<\omega$ there are $i<j<\omega$ such that
(a) $\bigcup_{l \neq n} W_{l} \cap[i, j]=\varnothing$;
(b) $2^{2^{2^{2^{j}}}}<\min \left[\bigcup_{l \neq n} W_{l} \backslash[0, i)\right]$;
(c) $2^{-2^{i}} \cdot \frac{1}{k}>\sum\left\{2^{-l}: l \in W_{n} \cap[i, j]\right\}$;
(d) There is $u \subseteq W_{n} \cap[i, j]$ satisfying $|u|>\left[2^{2^{2^{2^{i}}}}\right]^{m+1}$;
(e) $i>k$.
2. Definition. (i) $X \subseteq{ }^{\omega>} 2$ obeys $u \subseteq \omega$ if there exist $u_{X} \subseteq u, \rho_{n} \in{ }^{n} 2$ such that $X=\left\{\rho_{n}: n \in u_{X}\right\}$.
(ii) $X$ almost obeys $u$ if it obeys $u \cup\{0,1, \ldots, n\}$ for some $n$.
(iii) $\left({ }^{\omega} 2\right)^{[X]}=\bigcup_{\rho \in X}\left({ }^{\omega} 2\right)^{[\rho]}=\left\{\nu: \nu \in{ }^{\omega} 2 \wedge(\exists \rho)(\rho \in X \wedge \rho \subset \nu\}\right.$.
(iv) $X$ obeys $\bar{W}$ if it almost obeys each $\bigcup_{l>n} W_{l}$.

Let $L_{b} M_{s}$ denote Lebesgue measure on $2^{\omega}$.
3. Definition. $Q=Q(\bar{W})$ is defined as follows:
(I) A condition has the form $(t, X)$ where
(i) $t$ is a function from $n \geq 2$ to $\mathbb{Q} \cap[0,1)$ and $2^{4|\rho|} \cdot t(\rho)$ is an integer for all $\rho \in{ }^{n \geq 2}$;
(ii) $t(\eta)=t(\eta \wedge\langle 0\rangle)+t(\eta \wedge\langle 1\rangle)$ for $\eta \in^{n>} 2$;
(iii) for $\rho \in{ }^{n} 2, t(\rho)<L_{b} M_{s}\left(\left({ }^{\omega} 2\right)^{[\rho]} \backslash\left({ }^{\omega} 2\right)^{[X]}\right)$;
(iv) $X$ is a finite union of sets obeying $\bar{W}$.
(II) $\left(t_{1}, X_{1}\right) \leq\left(t_{2}, X_{2}\right)$ iff $t_{1} \subseteq t_{2}, X_{1} \subset X_{2}$, and if $\eta \in \operatorname{Dom}\left(t_{2}\right) \backslash \operatorname{Dom}\left(t_{1}\right)$ and $\eta \in X_{1}$, then $t_{2}(\eta)=0$.
4. Claim. $Q \vDash$ Souslin .
5. Claim. $Q$ ह ccc .

Proof. Let $\left\langle\left\langle t_{\alpha}, X_{\alpha}\right\rangle: \alpha<\omega_{1}\right\rangle$ be an $\omega_{1}$-sequence of members of $Q$. W.l.o.g. $t_{\alpha}=t_{\beta}$ for $\alpha \neq \beta<\omega_{1}$. Now let $r_{\alpha}^{\rho}$ be a positive rational satisfying

$$
r_{\alpha}^{\rho}<t_{\alpha}(\rho)-L_{b} M_{s}\left(\left(^{\omega} 2\right)^{[\rho]} \backslash\left({ }^{\omega} 2\right)^{[X]}\right) .
$$

Therefore w.l.o.g. $r_{\alpha}^{\rho}=r_{\beta}^{\rho}$ for $\alpha \neq \beta<\omega_{1}$. Let $m_{\rho}$ be such that $2^{-m_{\rho}}<$ $r_{\alpha}^{\rho} / 4^{|t|}$. Let $m=\max \left\{m_{\rho}: \rho \in \operatorname{Dom}\left(t_{\alpha}\right)\right\}$. Also w.l.o.g. we may assume that $X_{\alpha} \upharpoonright m=X_{\beta} \upharpoonright m$. Then it is not hard to see that $\left\langle\left\langle t_{\alpha}, X_{\alpha}\right\rangle: \alpha<\omega_{1}\right\rangle$ is a set of pairwise compatible members of $Q$.
6. Notation. Let $\operatorname{Dom}^{+}(t)=\{\eta \in \operatorname{Dom}(t): t(\eta)>0\}$. $\mathbf{T}=\bigcup\left\{\operatorname{Dom}^{+}(t):\langle t, X\rangle\right.$ $\left.\in \mathbf{G}_{Q}\right\}$ is a $Q$-name.
7. Claim. If $X_{n}$ obeys $W_{n}$ for $n<\omega$, each $X_{n}$ finite $\left\langle X_{n}: n<\omega\right\rangle \in V$, then in $V^{Q}$ the following hold:
(i) $\mathbf{T}$ is a perfect subset of ${ }^{\omega>} 2$.
(ii) $L_{b} M_{s}(\lim \mathbf{T})=t\langle 0\rangle$ for some (any) $\langle t, X\rangle \in G_{Q}$.
(iii) For some $k, \mathbf{T}$ is disjoint from $\bigcup_{n \geq k} X_{n}$.
(iv) Therefore, $\lim (\mathbf{T}) \cap\left({ }^{\omega} 2\right)^{\left\lfloor\bigcup_{n>k} X_{n}\right]}=\varnothing$.

## Proof. Clear.

Now we will introduce a technical device that we will use in order to build our models. After this we will show a theorem about finite support iteration forcing.
8. Definition. Let $\lambda$ be a cardinal, and $n$ an integer. We call $\bar{X}(\lambda, n)$-big if
(i) $\bar{X}$ is a family of subsets of ${ }^{\omega>} 2$ each one obeying $\bar{W}$;
(ii) If $X_{\zeta} \in \bar{X}$ for $\zeta<\lambda$ are pairwise distinct then for every $m, k$ there are [ $i, j$ ] and $u$ such that
(a) $\bigcup_{l \neq n} W_{l} \cap[i, j]=\varnothing$;
(b) $u \subseteq W_{n} \cap[i, j]$;
(c) $|u|>\left[2^{2^{2^{i}}}\right]^{m+1}$;
(d) $\min \left(\bigcup_{l \neq n} W_{l} \backslash[0, i]\right)>2^{2^{2^{2^{j}}}}$;
(e) $2^{-2^{i}} \cdot \frac{1}{k}>\sum_{l \in W_{n} \cap[i, j]} 2^{-l}$;
(f) for every $\left\langle\rho_{l}: l \in u\right\rangle \in \prod_{l \in u}{ }^{l} 2$ there is $\zeta$ such that for each $l \in u$, $X_{\zeta} \cap^{l} 2=\left\{\rho_{l}\right\} ;$
(g) $i>k$.

## 9. Lemma. Assume that

(i) $\bar{Q}=\left\langle P_{i}, Q_{j}: i \leq \alpha, j<\alpha\right\rangle$ is a finite support iteration and $\operatorname{cof}(\lambda)=$ $\lambda>\aleph_{0}, \lambda>\alpha=\bigcup \alpha \neq 0$.
(ii) $\bar{X}$ is $(\lambda, n)$-big in $V$.
(iii) For each $i<\alpha, \bar{X}$ is $(\lambda, n)$-big in $V^{P_{i}}$.

Then in $V^{P_{\alpha}}$ we also have that $\bar{X}$ is $(\lambda, n)$-big.
Proof. Trivial $(\lambda>\alpha!)$.
10. Lemma. $Q$ preserves " $\bar{X}$ is $(\lambda, n)$-big" when $\operatorname{cof}(\lambda)=\lambda>\kappa_{0}$.

Proof. Suppose that $p \Vdash_{Q}$ " $\left\langle\mathbf{X}_{\zeta}: \zeta<\lambda\right\rangle$ is a counterexample". Choose by induction on $\beta<\lambda, \zeta_{\beta}<\lambda, \mathbf{X}_{\zeta_{\beta}}, X_{\beta}, p_{\beta}$ such that $p \leq p_{\beta} \in Q$

$$
p_{\beta} \Vdash " \mathbf{X}_{\zeta_{\beta}}=X_{\beta} \wedge X_{\beta} \notin\left\{X_{\gamma}: \gamma<\beta\right\} " .
$$

Let $p_{\beta}=\left\langle t_{\beta}, Y_{\beta}\right\rangle$, w.l.o.g. $t_{\beta}=t, \operatorname{Dom}\left(t_{\beta}\right)=n^{*} \geq 2 Y_{\beta}=\bigcup_{l<k^{*}} Y_{\beta, l}$, and each $Y_{\beta, l} \backslash^{e \geq 2}$ obey $\bigcup_{l>n} W_{l}$. Let $k^{1}>k^{*}, k$ such that $4 n^{*}<k^{1}$, and let $m^{1}=m+1$. We know that $\bar{X}$ is $(\lambda, n)$-big in $V$; therefore, we can find $[i, j]$, $n$, and $\left\langle\zeta(\bar{\rho}): \bar{\rho} \in \prod_{l \in n}{ }^{l} 2\right\rangle$ satisfying conditions $8(\mathrm{i})$, (ii) for $k^{1}, m^{1}$.
11. Claim. There is $u_{1} \subseteq u$ such that $\left|u_{1}\right| \geq\left(2^{2^{2^{2^{i}}}}\right)^{m+1}$ and there is a function $H: \prod_{l \in u_{1}}{ }^{l} 2 \rightarrow \prod_{l \in u}{ }^{l} 2$ satisfying
(a) $(H(\bar{\rho})) \mid u_{1}=\bar{\rho}$;
(b) for every $\bar{\rho}_{1} \bar{\rho}_{2} \in \prod_{l \in u_{1}}{ }^{\prime} 2$ we have

$$
\left\langle Y_{\zeta\left(H\left(\bar{\rho}_{1}\right)\right), l} \cap{ }^{i>} 2: l<k^{*}\right\rangle=\left\langle Y_{\zeta\left(H\left(\bar{\rho}_{2}\right)\right), l} \cap^{i>} 2: l<k^{*}\right\rangle
$$

Proof. We know that $i>k^{1}>k^{*}$; therefore, $2^{2^{2^{2^{1}}}}$ is bigger than the number of possibles $\left\langle Y_{\zeta, l} \cap^{i>} 2: l<k^{*}\right\rangle$. This means that the function $G: \prod_{l \in u}{ }^{\prime 2} \rightarrow$ Range $(G)$, given by

$$
G(\bar{\rho})=\left\langle Y_{\zeta(\bar{\rho}), I} \cap^{i>} 2: i<k^{*}\right\rangle,
$$

satisfies $|\operatorname{Range}(\bar{\rho})| \leq 2^{2^{2^{2^{i}}}}$. On the other hand we have $|u|>\left(2^{2^{2^{2^{i}}}}\right)^{m^{1}+1}$ and $m^{1}=m+1$. From this we have that there are a sequence of disjoint sets $\left\langle u_{e}: e \in \operatorname{Range}(G)\right\rangle$, each $u_{e} \subseteq u$ satisfying

$$
\left|u_{e}\right| \geq\left(2^{2^{2^{2^{i}}}}\right)^{m+1}
$$

Fix an ordering of $\operatorname{Range}(G)$. By induction on $e \in \operatorname{Range}(G)$ we will try to pick $\bar{\rho}^{e}$ satisfying
(i) $\bar{\rho}^{e} \in \prod_{l \in u_{e}}{ }^{l} 2$;
(ii) If $\bar{\rho} \in \prod_{l \in u} l^{l} 2$ and $\bar{\rho}^{e}=\bar{\rho} \upharpoonright u_{e}$ and for each $e^{1}<e \bar{\rho}^{e^{1}}=\bar{\rho} \upharpoonright u_{e^{1}}$, then $G(\bar{\rho}) \neq e$. If we can do this induction then let $\rho^{*}=\bigcup_{e \in \operatorname{Range}(G)} \bar{\rho}^{e}$, and clearly $G\left(\rho^{*}\right) \neq e$ for every $e \in \operatorname{Range}(G)$-a contradiction.
Therefore there exists the first $e \in \operatorname{Range}(G)$ such that we cannot pick $\bar{\rho}^{e}$ satisfying (i) and (ii). This means that for each $\bar{\rho} \in \prod_{l \in u_{e}}{ }^{\prime} 2$ there is $v_{\rho} \in$ $\prod_{l \in u}{ }^{l} 2, \rho=v_{\rho} \upharpoonright u_{l}$, and $G\left(v_{\rho}\right)=e$. This clearly defines $H$.

Now we will finish with the proof of Lemma 10.
Let $u_{1}, H$ be given by the claim. It will be enough to show that

$$
\langle t, X\rangle=\left\langle t, \bigcup_{\substack{l<k^{*} \\ \bar{\rho} \in u_{1}}} Y_{\zeta(H(\bar{\rho})), l}\right\rangle
$$

is a condition.
By assumption on $u_{1}$ and $H$

$$
X \cap^{i>} 2=\bigcup_{l<k^{*}} Y_{\zeta(H(\bar{\rho})), l}
$$

(for some (any) $\rho \in \prod_{l \in u_{1}}{ }^{l} 2$ ).
We should check only $3(\mathrm{I})$ (iii). We know that for each $\eta \in t$

$$
t(\eta)>2^{-4 n^{*}}
$$

Also we know that

$$
t(\eta)-L_{b} M_{s}\left(\left({ }^{\omega} 2\right)^{[\eta]} \backslash\left({ }^{\omega} 2\right)^{[X]}\right)>0
$$

therefore, for each $\rho \in X \cap^{i>} 2$ and $\rho$ compatible with $\eta$

$$
t(\eta)-2^{-|\rho|}>0
$$

$$
2^{i-1}\left(t(\eta)-2^{-|\rho|}\right) \text { is an integer. }
$$

Therefore

$$
t(\eta)-L_{b} M_{s}\left(\left({ }^{\omega} 2\right)^{[\eta]} \backslash\left({ }^{\omega} 2\right)^{\left[X \cap^{i>} 2\right]}\right)>2^{-(i-1)}
$$

Now by assumption 8(e)

$$
L_{b} M_{s}\left(\left({ }^{\omega} 2\right)^{[\eta]} \cap\left({ }^{\omega} 2\right)^{\left[X \cap^{[i . j]_{2]}}\right.}\right)<2^{-2^{i}}
$$

and also

$$
L_{b} M_{l}\left(\left({ }^{\omega} 2\right)^{[\eta]} \cap\left({ }^{\omega} 2\right)^{\left[X \cap^{(\omega-\jmath} 2\right]}\right)<2^{-j}
$$

Thus

$$
t(\eta)-L_{b} M_{s}\left(\left({ }^{\omega} 2\right)^{[X]} \cap\left({ }^{\omega} 2\right)^{[\eta]}\right)>0
$$

Let $V \vDash C H$, and let $\bar{W}=\left\langle W_{n}: n<\omega\right\rangle$ satisfy the condition from 2, each $W$ is infinite. Let $\bar{Q}=\left\langle P_{i}, Q_{j}: i \leq \omega_{1}, j<\omega_{1}\right\rangle$ be a finite support iteration. $Q_{0}$ adds $\aleph_{2}$ Cohen reals. We interpret it as adding $\left\langle X_{\zeta, n}: \zeta<\omega_{2}, n<\omega\right\rangle$, where $X_{\zeta, n} \subseteq{ }^{\omega>2}$ obeys $W_{n}$. Each $Q_{i+1}$ is $Q$ from Definition 3 .
12. Claim. For each $n<\omega$

$$
V^{Q_{0}} \Vdash \text { " }\left\langle X_{\zeta, n}: \zeta<\omega_{2}\right\rangle \text { is }\left\langle\aleph_{2}, n\right\rangle \text {-big". }
$$

Proof. Clear using the properties of $\bar{W}$.
Therefore $V^{P_{\omega_{2}}} \vDash$ " $\left\langle X_{\zeta, n}: \zeta<\omega_{1}\right\rangle$ is $\left(\aleph_{2}, n\right)$-big for each $n "$. Let $\left\langle T_{i+1}: i<\right.$ $\left.\omega_{1}\right\rangle$ be the reals (perfect trees) given by the $Q_{i}$ 's. Clearly $\left\langle T_{i+1}: i<\omega_{1}\right\rangle$ is generic for an $\omega_{1}$-iteration of $Q$. This is an $\omega_{1}$-iteration of Souslin forcing satisfying ccc ; therefore, $\left\langle T_{i+1}: i<\omega_{1}\right\rangle$ is also generic over $V$. Let

$$
\begin{aligned}
& V^{a}=V\left[\left\langle T_{i+1}: i<\omega_{1}\right\rangle\right] \\
& V^{b}=V\left[\left\langle X_{\zeta, n}: \zeta<\omega_{2}, n<\omega\right\rangle\right]\left[\left\langle T_{i+1}: i<\omega_{1}\right\rangle\right]
\end{aligned}
$$

Then

$$
V^{a} \vDash C H, \quad V^{a} \text { is a class of } V^{b} .
$$

For each $\zeta<\aleph_{2}, X_{\zeta, n} \subseteq{ }^{\omega>} 2$, so $I_{\zeta, n}=\left\{\left({ }^{\omega} 2\right)^{[\rho]}: \rho \in X_{\zeta, n}\right\}$ is a subset of (Random) $)^{V^{a}}$ (but not in $V^{a}!!$ ).

We want to show that "for every large enough $\zeta, I_{\zeta, n}$ is a predense subset of (Random) $)^{V^{a}}$ ".

If $I_{\zeta, n}$ is a counterexample, then there are $\varepsilon>0$ and perfect $T_{\zeta, n} \subseteq{ }^{\omega} 2$ in $V^{a}$, such that $\left(\lim T_{\zeta, n}\right) \cap \bigcup I_{\zeta, n}=0$ and $L_{b} M_{s}\left(\lim T_{\zeta, n}\right)>\varepsilon$. But $V^{a} \vDash C H$, so for some $T, u=\left\{\zeta<\aleph_{2}: T_{\zeta, n}\right.$ is well defined and $\left.=T\right\}$ has cardinality $\aleph_{2}$.
13. Claim. $\left\{X_{\zeta, n}: \zeta \in u\right\}$ contradicts $\left\{X_{\zeta, \eta}: \zeta<\omega_{2}\right\}$ is ( $\aleph_{2}, n$ )-big.

Proof. Clearly $\bigcup_{\zeta \in \eta} X_{\zeta, n}$ contains ${ }^{l} 2$ for arbitrarily large $l$ (by using 8(e)). Therefore $T \cap^{l} 2=\varnothing$. But this contradicts $L_{b} M_{s}(\lim T)>0$.

So for each $n<\omega$ for every large enough $\zeta, I_{\zeta, n}$ is predense in (Random) ${ }^{V^{a}}$. So for some $\zeta$ this holds for every $n$ (really by homogeneity of forcing this holds for ever $\zeta$ and we can use $Q_{0}$ being one Cohen real).

Let $h_{n}: I_{\zeta, n} \rightarrow \omega$ be such that $h_{n}\left(\left({ }^{\omega} 2\right)^{\left[\rho_{1}\right]}\right)=h_{n}\left(\left({ }^{\omega} 2\right)^{\left[\rho_{2}\right]}\right)$ iff $\rho_{1}$ and $\rho_{2}$ are comparable. So $\left\langle h_{n}: n<\omega\right\rangle$ describes a (Random) ${ }^{V_{a}}$-name $\mathbf{h}$, namely, $\mathbf{h}(n)=h_{n}\left(\left({ }^{\omega} 2\right)^{[\rho]}\right)$ if $\left({ }^{\omega} 2\right)^{[\rho]} \in I_{\zeta, n}$ and $\left({ }^{\omega} 2\right)^{[\rho]} \in$ Generic set. This will be the name of generic real.

Let $B_{0} \in(\text { Random })^{V^{a}}, f: \omega \rightarrow \omega$ in $V^{b}$. We want to prove that for some $B_{1}, B_{0} \subset B_{1} \in(\text { Random })^{V^{a}}$ and $B_{1} \Vdash f<^{*} \mathbf{h}$. Let $Y_{n}=\left\{\rho \in X_{\zeta, n}: \rho\right.$ minimal in $X_{\zeta, n}$ (i.e., $\rho \upharpoonright l \notin X_{\zeta, n}$ for $\left.l<l g(l)\right)$ and $\left.h_{n}\left(\left({ }^{\omega} 2\right)^{[\rho]}\right) \leq f(n)\right\}$. Clearly $Y_{n}$ is finite and obeys $W_{n}\left(Y_{n} \subseteq X_{\zeta, n}\right)$. Therefore $\bigcup_{n<\omega} Y_{n}$ almost obeys each $\bigcup_{n>k} W_{n}$ for each $k$. Hence it obeys $\bar{W}$.

Also $\bigcup_{n<\omega} Y_{n} \in V^{P_{\omega_{1}}}$, so for some $i<\omega_{1}$, it belongs to $V^{P_{i}}$, w.l.o.g. $i>0$. Now for some $\rho \in{ }^{\omega>} 2, L_{b} M_{s}\left(\left({ }^{\omega} 2\right)^{[\rho]} \cap B_{0}\right) \geq 2^{-|\rho|}(1-1 / 100)$.

On the other hand, by genericity we can find $j \in\left[i, \omega_{1}\right)$ such that for some $k, t(\rho)=(1-1 / 100) 2^{-|\rho|}$ and $(t, \phi)$ is in the generic subset for $Q_{j}$. Therefore $T_{j+1}$ satisfies $L_{b} M_{s}\left(\lim T_{j+1} \cap\left({ }^{\omega} 2\right)^{[\rho]}\right)=(1-1 / 100) 2^{-|\rho|}$. So $B_{1}=$ $B_{0} \cap \lim T_{j+1} \geq B_{0}$ is a condition in Random $V^{a}$ (because $T_{j+1} \in V^{a}$ ). But it is also forced that for some $n, T_{i} \cap\left(\bigcup_{k>n} Y_{h}\right)=\varnothing$ by 7(iii). And this implies $B_{1} \Vdash f<^{*} \mathbf{h}$. This finishes the proof of the theorem.
Remark. Recently, Janusz Pawlikowski, motivated by this present work, showed that if you adjoin an "infinitely often equal" real and then force with the random algebra of the ground model, you get dominating reals (in fact Hechler-generic).

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Department of Mathematics, University of California Berkeley, Berkeley, California 94720

Current address: Department of Mathematics, Bar Ilan University, 52900 Ramat Gan, Israel
Institute of Mathematics, The Hebrew University, Jerusalem, Israel
Current address: Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903


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