

LINEAR TRANSFORMATIONS PRESERVING POTENT MATRICES

MATEJ BREŠAR AND PETER ŠEMRL

(Communicated by Maurice Auslander)

ABSTRACT. Linear transformations of M_n , the algebra of $n \times n$ matrices over \mathbb{C} , which preserve the set of all potent matrices, are characterized.

Let M_n be the algebra of $n \times n$ matrices over a field. A number of authors have characterized linear transformations θ on M_n which preserve some subsets Γ of M_n (i.e., $\theta(\Gamma) \subseteq \Gamma$). Let us list some examples of such subsets: the case when Γ is the set of all singular matrices was considered by Dieudonné [3]; $\Gamma = \{A \in M_n | \text{rank } A \leq 1\}$ by Jacob [7] and Marcus and Moyls [9]; Γ is a linear group by Dixon [4]; and Γ is the set of all nilpotent matrices by Botta, Pierce, and Watkins [1]. In a recent paper [2], motivated by a problem of characterizing local automorphisms and local derivations of some operator algebras (see, e.g., [8]), the present authors considered the case when Γ is the set of all projections in M_n . In this paper, we consider a more general situation; namely, we deal with linear transformations preserving potent matrices (recall that a matrix A is said to be potent if $A^r = A$ for some integer $r \geq 2$). The set of all potent matrices will be denoted by π . For any integer $r \geq 2$ we define $\pi_r = \{A \in M_n | A^r = A\}$. By A^t and $\text{tr}(A)$ we denote the transpose and the trace of A , respectively. The aim of this paper is to prove the following

Theorem. *Let M_n be the algebra of $n \times n$ matrices over the complex field \mathbb{C} , and let $\theta \neq 0$ be a linear transformation on M_n . The following conditions are equivalent.*

- (i) $\theta(\pi) \subseteq \pi$.
- (ii) *There exists an integer $r \geq 2$ such that $\theta(\pi_r) \subseteq \pi_r$.*
- (iii) θ is either of the form

$$\theta(A) = cUAU^{-1} \quad \text{or} \quad \theta(A) = cUA^tU^{-1},$$

where $U \in M_n$ is an invertible matrix and $c \in \mathbb{C}$ is a root of unity.

Proof. It is clear that (iii) implies (i) and (ii). We shall prove the converse implications.

First, assume that $\theta(\pi_r) \subseteq \pi_r$ for some $r \geq 2$, and let us show that (iii) holds. Let $P, Q \in M_n$ be orthogonal projections (i.e., $P^2 = P$, $Q^2 = Q$,

Received by the editors February 3, 1992.

1991 *Mathematics Subject Classification.* Primary 15A04.

Key words and phrases. Linear transformation, potent matrix.

and $PQ = QP = 0$). Let $\lambda_1, \lambda_2, \dots, \lambda_{r-1}$ be $(r-1)$ -roots of unity, and note that $P + \lambda_i Q \in \pi_r$, $i = 1, 2, \dots, r-1$. By the assumption it follows that $\theta(P + \lambda_i Q)^r = \theta(P + \lambda_i Q)$; that is,

$$(A + \lambda_i B)^r = A + \lambda_i B,$$

where $A = \theta(P)$ and $B = \theta(Q)$. As $A^r = A$ and $B^r = B$ (namely, P and Q belong to π_r), this relation can be written in the form

$$\lambda_i C_1 + \lambda_i^2 C_2 + \dots + \lambda_i^{r-1} C_{r-1} = 0, \quad i = 1, 2, \dots, r-1,$$

where

$$C_1 = A^{r-1}B + A^{r-2}BA + \dots + AB A^{r-2} + B A^{r-1},$$

$$C_2 = A^{r-2}B^2 + A^{r-3}BAB + \dots + B^2 A^{r-2},$$

\vdots

$$C_{r-1} = AB^{r-1} + BAB^{r-2} + \dots + B^{r-2}AB + B^{r-1}A.$$

Since the λ_i 's are nonzero and mutually different, it follows that $C_1 = C_2 = \dots = C_{r-1} = 0$; thus, in particular,

$$A^{r-1}B + A^{r-2}BA + \dots + AB A^{r-2} + B A^{r-1} = 0.$$

Multiply this relation first from the left by A and then from the right by A . Comparing the two relations so obtained and using $A^r = A$, we get that A and B commute. Hence $A^{r-1}B = 0$ and, therefore, $AB = A^r B = 0$.

Thus, we proved the following: If P and Q are orthogonal projections then $\theta(P)\theta(Q) = \theta(Q)\theta(P) = 0$. Now pick a selfadjoint matrix S . There exist mutually orthogonal projections P_1, P_2, \dots, P_n and real numbers t_1, t_2, \dots, t_n such that $S = \sum_{i=1}^n t_i P_i$. Since $\theta(P_i)\theta(P_j) = \theta(P_j)\theta(P_i) = 0$, $i \neq j$, and $\theta(P_i)^r = \theta(P_i)$, it follows that

$$\theta(S^r) = \sum_{i=1}^n t_i^r \theta(P_i) = \theta(S)^r.$$

Set $K = \theta(I)$. In the relation $\theta(S^r) = \theta(S)^r$, replace S by $S + tI$, where t is a real number. Then we get

$$\begin{aligned} & \theta(S^r) + rt\theta(S^{r-1}) + \dots + (r(r-1)/2)t^2\theta(S^2) + rt^{r-1}\theta(S) + t^r K \\ &= \theta((S + tI)^r) = (\theta(S) + tK)^r \\ &= \theta(S)^r + t(\theta(S)^{r-1}K + \theta(S)^{r-2}K\theta(S) + \dots + K\theta(S)^{r-1}) \\ &+ \dots + t^{r-2}(\theta(S)^2K^{r-2} + \theta(S)K\theta(S)K^{r-3} + \dots + K^{r-2}\theta(S)^2) \\ &+ t^{r-1}(\theta(S)K^{r-1} + K\theta(S)K^{r-2} + \dots + K^{r-1}\theta(S)) + t^r K^r. \end{aligned}$$

Comparing the coefficients at t^{r-1} , we obtain

$$r\theta(S) = \theta(S)K^{r-1} + K\theta(S)K^{r-2} + \dots + K^{r-1}\theta(S).$$

Multiplying this relation first from the left by K and then from the right by K and using $K^r = K$, it follows that $\theta(S)K = K\theta(S)$. Consequently, comparing the coefficients at t^{r-2} , we arrive at

$$\theta(S^2) = K^{r-2}\theta(S)^2,$$

where S is an arbitrary selfadjoint matrix. Define a transformation φ on M_n by

$$\varphi(A) = K^{r-2}\theta(A).$$

For a selfadjoint $S \in M_n$ we then have

$$\varphi(S^2) = K^{r-2}\theta(S^2) = K^{2r-4}\theta(S)^2,$$

and, since $\theta(S)$ commutes with K ,

$$\varphi(S)^2 = (K^{r-2}\theta(S))^2 = K^{2r-4}\theta(S)^2.$$

Thus $\varphi(S^2) = \varphi(S)^2$. Replacing S by $S + T$, where S and T are selfadjoint, we then get

$$\varphi(ST + TS) = \varphi(S)\varphi(T) + \varphi(T)\varphi(S).$$

Since every $X \in M_n$ can be written in the form $X = S_1 + iS_2$ with S_1, S_2 selfadjoint, it follows that

$$\varphi(XY + YX) = \varphi(X)\varphi(Y) + \varphi(Y)\varphi(X)$$

holds for all $X, Y \in M_n$; thus, φ is a Jordan homomorphism. We claim that either φ is nonsingular or $\varphi = 0$. Indeed, $\text{Ker } \varphi$ is a Jordan ideal (i.e., $\text{Ker } \varphi$ is a linear subspace such that $A \in \text{Ker } \varphi$ implies $AX + XA \in \text{Ker } \varphi$ for every $X \in M_n$), but then $\text{Ker } \varphi = M_n$ or $\text{Ker } \varphi = \{0\}$ (cf. [6, Theorem 1.1]). Suppose $\varphi = 0$. Then we have $0 = \varphi(I) = K^{r-2}\theta(I) = K^{r-1}$ and, therefore, $K = K^r = 0$. We showed that for any projection P we have $\theta(P)\theta(I - P) = 0$, so $K = 0$ gives $\theta(P)^2 = 0$, and thus $\theta(P) = \theta(P)^r = 0$. Since every matrix can be written as a linear combination of projections, it follows that $\theta = 0$, contrary to the assumption. Thus $\varphi \neq 0$, so φ is nonsingular; but then φ is either an automorphism or an antiautomorphism [6, Theorem 3.1]. It is well known that in the first case it is of the form $\varphi(A) = UAU^{-1}$ for some invertible U in M_n , and in the second case, $\varphi(A) = UA^tU^{-1}$.

Recall that $\varphi(A) = K^{r-2}\theta(A)$. As φ is nonsingular, θ must be nonsingular, too. We proved that K commutes with $\theta(S)$ for every selfadjoint matrix S ; this clearly yields that K commutes with $\theta(A)$ for any $A \in M_n$. Since θ is nonsingular, it follows that $K = cI$ for some $c \in \mathbb{C}$. Using $K^r = K$ and $K \neq 0$ we get $c^{r-1} = 1$; hence, $\theta(A) = c\varphi(A)$. The proof of the assertion that (ii) implies (iii) is thereby completed.

Now we come to the central part of the proof; namely, we shall prove that (i) implies (iii). Let us first point out two simple observations which will be used repeatedly. A matrix A is potent if and only if A is diagonalizable and its eigenvalues are either roots of unity or 0. Therefore, every upper triangular matrix with mutually different roots of unity on the diagonal is potent.

We divide the proof into eight steps.

Step 1. If $N \in M_n$ is nilpotent then $\theta(N)$ is nilpotent.

Proof of Step 1. There exists an invertible $S \in M_n$ such that SNS^{-1} is strictly upper triangular. Let R be a diagonal matrix with mutually different roots of unity on a diagonal. Then $R + \alpha SNS^{-1}$ is a potent matrix for any $\alpha \in \mathbb{C}$. Therefore, the same is true for matrices $R_1 + \alpha N$, $\alpha \in \mathbb{C}$, where $R_1 = S^{-1}RS$; that is, for any $\alpha \in \mathbb{C}$ there exists an integer $n_\alpha \geq 2$ such that

$$(\theta(R_1) + \alpha\theta(N))^{n_\alpha} = \theta(R_1) + \alpha\theta(N).$$

Clearly, there exists an integer $n_0 \geq 2$ such that

$$(\theta(R_1) + \alpha\theta(N))^{n_0} = \theta(R_1) + \alpha\theta(N)$$

holds for infinitely many α , but then this must be fulfilled for every $\alpha \in \mathbb{C}$; thus, we have

$$\begin{aligned} \theta(N)^{n_0} &= \lim_{|\alpha| \rightarrow \infty} (\alpha^{-1}\theta(R_1) + \theta(N))^{n_0} \\ &= \lim_{|\alpha| \rightarrow \infty} (\alpha^{-n_0}\theta(R_1) + \alpha^{-n_0+1}\theta(N)) = 0. \end{aligned}$$

Step 2. There exists $c \in \mathbb{C}$ such that $\text{tr}(\theta(A)) = c \text{tr}(A)$.

Proof of Step 2. It is easy to see that the linear span of all nilpotent matrices is the space sl_n of matrices with trace zero (namely, denoting by E_{ij} the matrix whose only nonzero entry is 1 in a position (i, j) , we see that the nilpotent matrices E_{ij} and $E_{ii} + E_{ij} - E_{ji} - E_{jj}$ for $i \neq j$ span sl_n). In view of Step 1, we then have $\theta(\text{sl}_n) \subseteq \text{sl}_n$. Since for any $A \in M_n$ the matrix $A - (\text{tr}(A)/n)I$ belongs to sl_n , it follows that $\text{tr}(\theta(A - (\text{tr}(A)/n)I)) = 0$; that is, $\text{tr}(\theta(A)) = c \text{tr}(A)$, where $c = \text{tr}(\theta(I))/n$.

Step 3. $\theta(I) \neq 0$.

Proof of Step 3. Suppose $\theta(I) = 0$. Let P be a projection. Since $I - P$ and $I - 2P$ are both potents, it follows that $-\theta(P) = \theta(I - P)$ and $-2\theta(P) = \theta(I - 2P)$ are potents too; but, then $\theta(P) = 0$. Since the linear span of all projections is M_n , it follows, contrary to the assumption, that $\theta = 0$.

Step 4. If $A \in \pi$ and $A \neq 0$, then $\theta(A) \neq 0$.

Proof of Step 4. Since $A \in \pi$, we have $A = R(\sum_{i=1}^k \lambda_i E_{ii})R^{-1}$, where $1 \leq k \leq n$, λ_i 's are roots of unity, and R is an invertible matrix. Suppose that $\theta(A) = 0$. Obviously, for any $i \in \{1, 2, \dots, k\}$ the matrices $A - R(\lambda_i E_{ii})R^{-1}$ and $A - 2R(\lambda_i E_{ii})R^{-1}$ are potents, and, therefore, $-\theta(R(\lambda_i E_{ii})R^{-1})$ and $-2\theta(R(\lambda_i E_{ii})R^{-1})$ are potents as well; but, then $\theta(R(\lambda_i E_{ii})R^{-1}) = 0$. Thus, we proved that $\theta(RE_{ii}R^{-1}) = 0$ for every $i \in \{1, 2, \dots, k\}$.

Now pick $j \in \{k+1, \dots, n\}$. The matrices $R(E_{11} + E_{1j})R^{-1}$ and $R(E_{11} + E_{j1})R^{-1}$ are projections, so it follows that $\theta(R(E_{11} + E_{1j})R^{-1}) = \theta(RE_{1j}R^{-1})$ and $\theta(RE_{j1}R^{-1})$ are potents; however, as $RE_{1j}R^{-1}$ and $RE_{j1}R^{-1}$ are nilpotents, $\theta(RE_{1j}R^{-1})$ and $\theta(RE_{j1}R^{-1})$ are nilpotents too (Step 1). Thus, $\theta(RE_{1j}R^{-1}) = \theta(RE_{j1}R^{-1}) = 0$. Since $R(E_{11} - E_{1j} + E_{j1} - E_{jj})R^{-1}$ is nilpotent, we have that

$$\theta(R(E_{11} - E_{1j} + E_{j1} - E_{jj})R^{-1}) = -\theta(RE_{jj}R^{-1})$$

is nilpotent; however, $\theta(RE_{jj}R^{-1})$ is also a potent since $RE_{jj}R^{-1}$ is a potent; hence, $\theta(RE_{jj}R^{-1}) = 0$. Thus we proved that $\theta(RE_{ii}R^{-1}) = 0$ for every $i \in \{1, 2, \dots, n\}$, which clearly contradicts the assertion of Step 3.

Step 5. If N is a nilpotent and $N \neq 0$, then $\theta(N) \neq 0$.

Proof of Step 5. It can be easily shown that there exists a nilpotent matrix M such that $M + N$ is a nonzero potent (for instance, if $N = E_{12} + E_{23} + \dots + E_{k-1,k}$, then these conditions are satisfied by $M = E_{k1}$; a general case then

follows by using a Jordan form of N). Thus $\theta(M + N)$ is a nonzero potent by Step 4; hence, $\theta(N)$ cannot be zero, for otherwise it would follow that $\theta(M + N) = \theta(M)$ is a nilpotent (Step 1).

Step 6. Let Ω be the algebra of upper triangular matrices. There exists an invertible matrix T such that $\theta(\Omega) \subseteq T\Omega T^{-1}$.

Proof of Step 6. Let Ω_0 be the algebra of strictly upper triangular matrices. Ω_0 is the space of nilpotents and its dimension is $n(n - 1)/2$; by Steps 1 and 5, the same is true for $\theta(\Omega_0)$. Therefore, by a result of Gerstenhaber [5] there exists an invertible $T \in M_n$ such that $\theta(\Omega_0) = T\Omega_0 T^{-1}$. The assertion will be proved by showing that θ maps diagonal matrices into the space $T\Omega T^{-1}$; thus, we must show that $T^{-1}\theta(A)T$ is an upper triangular matrix for any diagonal matrix A . It suffices to consider the case when A is potent and its entries are mutually different (namely, the set of such matrices spans the space of diagonal matrices). For any $N \in \Omega_0$ we have $A + N \in \pi$. Hence $\theta(A) + \theta(N) \in \pi$, so $T^{-1}\theta(A)T + T^{-1}\theta(N)T \in \pi$. Since N is an arbitrary matrix in Ω_0 and $\theta(\Omega_0) = T\Omega_0 T^{-1}$, it follows that $T^{-1}\theta(A)T + \Omega_0 \subseteq \pi$. Using standard arguments one shows that this implies that the matrix $T^{-1}\theta(A)T$ is upper triangular.

Step 7. $\theta(I) = cI$, and c is a root of unity.

Proof of Step 7. Define $\sigma: M_n \rightarrow M_n$ by $\sigma(A) = T^{-1}\theta(A)T$. Obviously, σ preserves potents, and, by the previous step, $\sigma(\Omega) \subseteq \Omega$.

Pick integers i, j , $1 \leq i, j \leq n$, $i \neq j$. Since E_{ii} , E_{jj} , $E_{ii} + E_{jj}$, and $E_{ii} - E_{jj}$ are potent and upper triangular matrices, the same is true for the matrices $A_i = \sigma(E_{ii})$, $A_j = \sigma(E_{jj})$, $A_i + A_j$, and $A_i - A_j$. Thus, their diagonal entries are either roots of unity or zero. This clearly yields that A_i and A_j cannot simultaneously have nonzero entries in positions (k, k) , $1 \leq k \leq n$. However, since the A_i 's are nonzero potents, each of them has at least one nonzero entry on a diagonal; thus, it has exactly one. By Step 2 it follows that this entry equals c for any i . Thus, we proved that for any $i \in \{1, 2, \dots, n\}$ there exists an integer $k(i) \in \{1, 2, \dots, n\}$ and a matrix $N_i \in \Omega_0$ such that $\sigma(E_{ii}) = cE_{k(i), k(i)} + N_i$. Note also that $k(i) \neq k(j)$ if $i \neq j$. Hence,

$$\sigma(I) = \sigma\left(\sum_{i=1}^n E_{ii}\right) = \sum_{i=1}^n (cE_{k(i), k(i)} + N_i) = cI + N,$$

where $N \in \Omega_0$. Since $\sigma(I)$ is potent, it follows easily that $N = 0$ and c is a root of unity. Finally, note that $\sigma(I) = cI$ implies $\theta(I) = cI$.

Define a mapping $\psi: M_n \rightarrow M_n$ by $\psi(A) = c^{-1}\theta(A)$. As we have proved that (ii) implies (iii), the proof of the theorem will be complete by showing that $\psi(\pi_2) \subseteq \pi_2$. Thus, our last step is

Step 8. ψ preserves projections.

Proof of Step 8. By Step 7, $\psi(I) = I$ and ψ preserves potents. Pick a projection P , and let us show that $\psi(P)$ is a projection. Since $\psi(P)$ is a potent, it suffices to show that an arbitrary eigenvalue λ of $\psi(P)$ is either 1 or 0. Since P , $I - P$, and $I - 2P$ are potents, $\psi(P)$, $I - \psi(P)$, and $I - 2\psi(P)$ are potents too, so

it follows that each of the numbers λ , $1 - \lambda$, and $1 - 2\lambda$ is either a root of unity or zero. The only two possibilities are that λ is either 1 or 0. The proof is thus complete.

REFERENCES

1. P. Botta, S. Pierce, and W. Watkins, *Linear transformations that preserve the nilpotent matrices*, Pacific J. Math. **104** (1983), 39–46.
2. M. Brešar and P. Šemrl, *Mappings which preserve idempotents, local automorphisms, and local derivations*, Canad. J. Math. (to appear).
3. J. Dieudonné, *Sur une généralisation du groupe orthogonal à quatre variables*, Arch. Math. **1** (1948), 282–287.
4. J. Dixon, *Rigid embeddings of simple groups in the general linear group*, Canad. J. Math. **29** (1977), 384–391.
5. M. Gerstenhaber, *On nilalgebras and linear varieties of nilpotent matrices I*, Amer. J. Math. **80** (1958), 614–622.
6. I. N. Herstein, *Topics in ring theory*, Univ. of Chicago Press, Chicago, 1969.
7. H. G. Jacob, *Coherence invariant mappings on Kronecker products*, Amer. J. Math. **77** (1955), 177–189.
8. D. R. Larson and A. R. Sourour, *Local derivations and local automorphisms of $B(X)$* , Proc. Sympos. Pure Math., vol. 51, part 2, Amer. Math. Soc., Providence, RI, 1990, pp. 187–194.
9. M. Marcus and B. Moys, *Transformations on tensor product spaces*, Pacific J. Math. **9** (1959), 1215–1221.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARIBOR, PF, KOROŠKA 160, 62000 MARIBOR, SLOVENIA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LJUBLJANA, JADRANSKA 19, 61000 LJUBLJANA, SLOVENIA