PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 119, Number 1, September 1993

POTENTIAL SPACE ESTIMATES FOR GREEN POTENTIALS IN CONVEX DOMAINS

STEPHEN J. FROMM

(Communicated by Barbara L. Keyfitz)

ABSTRACT. Weak type $(1, 1)$ bounds are demonstrated for the operators

$$f \mapsto \int \nabla_x \nabla_x G(x, y) f(y)\, dy \quad \text{and} \quad f \mapsto \int \nabla_x \nabla_y G(x, y) f(y)\, dy,$$

where G is the Green operator for the Dirichlet problem for the Poisson equation on a bounded convex domain in $\mathbb{R}^n$. These results are used to investigate smoothing properties of the Green operator in potential spaces. An application is given to the restriction of the potential space to the boundary of the domain.

0. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded convex domain. Consider the boundary value problem

$$\left\{ \begin{array}{ll} \triangle u = f & \text{in } \Omega, \\ u \,|_{\partial\Omega} = 0. \end{array} \right. \tag{1}$$

(The boundary condition on u is taken to mean that $u \in W_0^{1,p}(\Omega)$ for some p, $1 < p < \infty$; see Corollary 1.) By the Dirichlet principle, given $f \in L_{-1}^2(\Omega)$ there exists a unique $u \in W_0^{1,2}(\Omega)$ satisfying (1), and the map $f \mapsto u$ is a Hilbert space isomorphism from $L_{-1}^2(\Omega)$ onto $W_0^{1,2}(\Omega)$ (see [Ada] for the definitions of these function spaces). We can then define Green's function $G(x, y)$ (the fundamental solution of (1)) to be the distribution kernel of the map $f \mapsto u$. Define

$$Gf(x) = \int_\Omega G(x, y) f(y)\, dy.$$

Let $|\cdot|$ denote Lebesgue measure on $\mathbb{R}^n$ and let ∇ and ∇^2 denote the first and second gradient operators, respectively. Denote the diameter of Ω by d. The main result of this paper is

Received by the editors September 9, 1991 and, in revised form, February 3, 1992.
1991 *Mathematics Subject Classification*. Primary 35J25, 35B65; Secondary 35J05.
Key words and phrases. Green potential, convex domain, potential spaces.
The author was supported in part by NSF grants DMS 84-51770 and DMS 88-04582. This work will appear later as part of the author's doctoral dissertation.

Theorem. *Let $f \in L^1(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is bounded and convex. Then for all $\lambda > 0$,*

$$|\{x \in \Omega : |\nabla^2(Gf)(x)| > \lambda\}| \leq \frac{C(n)}{\lambda}\|f\|_{L^1(\Omega)}, \tag{2}$$

$$\left|\left\{x \in \Omega : \left|\int_\Omega \nabla_x \nabla_y G(x, y) f(y)\, dy\right| > \lambda\right\}\right| \leq \frac{C(d, n)}{\lambda}\|f\|_{L^1(\Omega)}. \tag{3}$$

The weak-type estimate (2) is an unpublished result due to Dahlberg, Verchota, and Wolff. It is a straightforward consequence of the analogous L^2 result (Lemma 1, due to Kadlec) and pointwise estimates of the gradients of $G(x, y)$. (Adolfsson has recently found a similar Hardy space result; see [Ado1].)

The weak-type estimate (3) is new. Its proof also relies on its L^2 analogue (Lemma 2, essentially the Dirichlet principle) and estimates of gradients of $G(x, y)$ but is substantively different from the proof of (2). (Whereas the standard Calderon-Zygmund-Hörmander bound (10) for singular kernels is the main step in the proof of (2), the analogous inequality for (3) fails for the planar domain constructed in §5.) Interpolation of (2), (3), and the L^2 results gives the estimates stated as Corollary 1. These estimates have a further application to the restriction of $L^p_{1+1/p}(\Omega)$ to $\partial\Omega$ when Ω is convex (see §4).

For $s \in \mathbb{R}$ and $1 < p < \infty$, let $L^p_s(\Omega)$ be the (Bessel) potential space in Ω, with norm $\|\cdot\|_{s,p}$. $L^p_s(\Omega)$ coincides with the usual Sobolev space $W^{s,p}(\Omega)$ when $s \in \mathbb{Z}$ and is defined for $s \in \mathbb{R}$ by the complex interpolation method (see [Ada]).

Corollary 1. *If $\Omega \in \mathbb{R}^n$ is bounded and convex, and if $f \in L^p_s(\Omega)$, then there is a unique $u \in W^{1,p}_0(\Omega) \cap L^p_{s+2}(\Omega)$ satisfying $\triangle u = f$, and*

$$\|u\|_{s+2,p} \leq C(d, s, p, n)\|f\|_{s,p} \tag{4}$$

for $-1 \leq s \leq 0$ and $1 < p < \frac{2}{s+1}$ (defining $\frac{2}{0} = \infty$) and for $s = 0$, $p = 2$.

Below, $\Omega \subseteq \mathbb{R}^n$ will be a bounded convex domain unless noted otherwise. C will denote a constant whose value may change between lines and within a line. C may depend on n but not on d, s, or p unless so indicated, and C may depend on Ω only in that it may depend on d. $L^p(\Omega)$, $L^p_s(\Omega)$, and so on will denote the obvious function spaces for vector- and tensor-valued functions.

Note Ω is a Lipschitz domain but generally not a C^1 domain.

1. Proof of (2)

First we state the analogous L^2 result. It follows from an integration by parts and the positivity of the second fundamental form of $\partial\Omega$ (see [G] for details).

Lemma 1. $\|\nabla^2(Gf)\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}$.

We now state some pointwise estimates on $G(x, y)$. Define

$$\delta(x) = \mathrm{dist}(x, \partial\Omega).$$

Proposition 1. *If* Ω *is convex* (*but not necessarily bounded*), *then*

$$|G(x, y)| \leq C\delta(y)|x-y|^{1-n}, \tag{5}$$

$$|\nabla_x G(x, y)| \leq C\delta(y)|x-y|^{-n}, \tag{6}$$

$$|\nabla_x G(x, y)| \leq C|x-y|^{1-n}, \tag{7}$$

$$|\nabla_x \nabla_y G(x, y)| \leq C|x-y|^{-n}, \tag{8}$$

$$|\nabla_x^2 \nabla_y G(x, y)| \leq C|x-y|^{-n}/[\min(|x-y|, \delta(x))]. \tag{9}$$

For $n \geq 3$, (5)–(8) can be found in [GW]; (9) follows easily from (8) by the techniques used there. When $n = 2$, only minor modifications need to be made in the proofs.

Consider now the operator $f \mapsto \nabla^2(Gf)$, which by Lemma 1 is bounded from $L^2(\Omega)$ to $L^2(\Omega)$. It may be extended to an operator K on functions on $\mathbb{R}^n$ with kernel $K(x, y) = \nabla_x^2 G(x, y)$ for x and $y \in \Omega$ and $K(x, y) = 0$ if x or $y \notin \Omega$. To prove the weak-type estimate above it suffices to show the Hörmander condition

$$\int_{|x-y_0|\geq 5r} |K(x, y) - K(x, y_0)|\, dx \leq C \tag{10}$$

for $|y - y_0| \leq r$ (and C depending on neither $y_0 \in \mathbb{R}^n$ nor $r > 0$).

First assume $y_0 \in \Omega$ and consider only those x in $\{x \in \Omega : |x - y_0| \geq 5r\}$ satisfying $|x - y_0| \leq \frac{1}{2}\delta(y_0)$. (This also constrains the choice of r.) Then

$$\begin{aligned}&|\nabla_x^2 G(x, y) - \nabla_x^2 G(x, y_0)| \\ &\quad \leq |y - y_0| \sup_{0\leq\lambda\leq 1} \{|\nabla_{y'} \nabla_x^2 G(x, y')| : y' = \lambda y + (1-\lambda) y_0\}.\end{aligned}$$

Since $|x - y'| \leq \frac{6}{5}\delta(x)$ and $|x - y_0| \leq \frac{5}{4}|x - y'|$, (9) gives

$$|\nabla_{y'} \nabla_x^2 G(x, y')| \leq C|x - y'|^{-n-1} \leq C|x - y_0|^{-n-1}.$$

The left-hand side of (10) (with domain of integration further restricted as above) is bounded by

$$C \int_{|x-y_0|\geq 5r} r|x - y_0|^{-n-1} dx \leq C,$$

completing this case.

Now consider the remaining cases: (a) $y_0 \notin \Omega$, and (b) $y_0 \in \Omega$, $|x - y_0| \geq \frac{1}{2}\delta(y_0)$, which may be rewritten together as $|x - y_0| \geq \frac{1}{2}\operatorname{dist}(y_0, \Omega^c)$ (where ^{c} denotes set complement in $\mathbb{R}^n$). Since $|x - y_0| \geq 5r$ and $|y - y_0| \leq r$, $|x - y| \geq \frac{1}{4}\operatorname{dist}(y, \Omega^c)$. Therefore for $|y - y_0| \leq r$ the integral in x of $|K(x, y) - K(x, y_0)|$ over $\{|x - y_0| \geq \max(5r, \frac{1}{2}\operatorname{dist}(y_0, \Omega^c))\}$ is bounded by the integral of $|K(x, y_0)|$ over $\{|x - y_0| \geq \frac{1}{2}\operatorname{dist}(y_0, \Omega^c)\}$ plus that of $|K(x, y)|$ over $\{|x - y| \geq \frac{1}{4}\operatorname{dist}(y, \Omega^c)\}$. Each of these is bounded by

$$\sup_{y\in\Omega} \int_{x\in\Omega,\, |x-y|\geq\delta(y)/4} |\nabla_x^2 G(x, y)|\, dx \tag{11}$$

since $K(x, y) = 0$ for x or $y \notin \Omega$. To complete the proof it suffices to show the finiteness of (11).

Take $\phi \in C_0^\infty(\mathbb{R}^+)$ with $\phi(t) = 0$ for $t \leq \frac{1}{2}$ or $t \geq 4$, and $\phi(t) = 1$ for $1 \leq t \leq 2$. Put $\phi_j(x) = \phi(|x-y|/[2^j\delta(y)])$. Then by the Cauchy-Schwarz inequality,

$$\int_{x\in\Omega,\,|x-y|\geq\delta(y)/4} |\nabla_x^2 G(x,y)|\,dx \leq \sum_{j=-2}^{\infty}\int_\Omega |\nabla_x^2[\phi_j(x)G(x,y)]|\,dx$$
$$\leq C\sum_{j=-2}^{\infty}[2^j\delta(y)]^{n/2}\left(\int_\Omega |\nabla_x^2[\phi_j(x)G(x,y)]|^2\,dx\right)^{1/2}.$$

Since $\phi_j(x)G(x, y)$ vanishes when $x \in \partial\Omega$, $\phi_j(\cdot)G(\cdot, y) = G[\Delta(\phi_j(\cdot)G(\cdot, y))]$. We may apply Lemma 1 to obtain

$$\int_\Omega |\nabla_x^2[\phi_j(x)G(x,y)]|^2\,dx \leq \int_\Omega |\Delta[\phi_j(x)G(x,y)]|^2\,dx.$$

Note $|\Delta_x[\phi_j(x)G(x,y)]|^2 \leq 2\{G(x,y)^2|\Delta\phi_j(x)|^2 + 4|\nabla_x G(x,y)|^2|\nabla\phi_j(x)|^2\}$, $|\Delta\phi_j| \leq C[2^j\delta(y)]^{-2}$, and $|\nabla\phi_j| \leq C[2^j\delta(y)]^{-1}$. (5) implies that, for $x \in \operatorname{supp}(\Delta\phi_j)$, $G(x,y) \leq C\delta(y)|x-y|^{1-n} \leq C\delta(y)[2^j\delta(y)]^{1-n}$; similarly, (6) implies that for $x \in \operatorname{supp}(\nabla\phi_j)$, $|\nabla_x G(x,y)| \leq C\delta(y)[2^j\delta(y)]^{-n}$. We find that

$$\int_\Omega |\nabla_x^2[\phi_j(x)G(x,y)]|^2\,dx \leq C2^{-2j}[2^j\delta(y)]^{-n}.$$

So

$$\int_{x\in\Omega,\,|x-y|\geq\delta(y)/4} |\nabla_x^2 G(x,y)|\,dx \leq C\sum_{j=-2}^{\infty}2^{-j} \leq C,$$

finishing the proof of (2).

Remark. Let $a(y)$ be supported in a ball $B = B(y_0, r) \subseteq \mathbb{R}^n$, with $\|a\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{|B|}$ and $\int a(y)\,dy = 0$. Then

$$\int_{\mathbb{R}^n} |Ka(x)|\,dx \leq \int_{B(y_0,5r)} |Ka(x)|\,dx$$
$$+\int_B\left(\int_{|x-y_0|\geq 5r}|K(x,y)-K(x,y_0)|\,dx\right)|a(y)|\,dy.$$

The first term on the right-hand side is bounded by using the Cauchy-Schwarz inequality and Lemma 1; the second term is bounded using (10). We find that

$$\int |Ka(x)|\,dx \leq C; \tag{12}$$

this is the main result of [Ado1]. Note that in [Ado1] the property $\int a\,dy = 0$ is not needed when $\operatorname{dist}(B, \Omega^c) \leq C\operatorname{diam}(B)$. This is true here as well because of the finiteness of (11). Note also that in [Ado1], (12) is proved in the case of Ω convex, unbounded, and lying above a Lipschitz graph of Lipschitz constant M; the constant in [Ado1] corresponding to the right-hand side of (12) depends on M. When applied to the case of bounded Ω, this gives atomic and L^p estimates with constants depending on the eccentricity of the domain, which is not the case here.

2. Proof of (3)

As before, we begin with the analogous L^2 result.

Lemma 2. $\| \int_\Omega \nabla_x \nabla_y G(x, y) f(y)\, dy\|_{L^2(\Omega; dx)} \le C(d)\|f\|_{L^2(\Omega)}$.

Proof. Since $G(x, y)$ vanishes as y tends to $\partial\Omega$, we may write

$$\int_\Omega \nabla_x \nabla_y G(x, y) f(y)\, dy = -\nabla_x \int_\Omega G(x, y) \nabla_y f(y)\, dy = -\nabla G(\nabla f)(x).$$

By the Dirichlet principle,

$$\|\nabla G(\nabla f)\|_{L^2(\Omega)} \le \|G(\nabla f)\|_{1,2} \le C(d)\|\nabla f\|_{-1,2}.$$

The lemma follows, since $\|\nabla f\|_{-1,2} \le \|f\|_{L^2(\Omega)}$.

Now fix $\lambda > 0$. Let $K(x, y) = \nabla_x \nabla_y G(x, y)$ if $x, y \in \Omega$, and $K(x, y) = 0$ otherwise; denote the operator corresponding to this kernel by K also.

Extend f to an element of $L^1(\mathbb{R}^n)$ by putting $f = 0$ outside of Ω. By the Calderon-Zygmund decomposition (see [St]), there are cubes Q_j, $j \in \mathbb{N}$, with disjoint interiors such that $|f(x)| \le \lambda$ a.e. in $\left(\bigcup_1^\infty Q_j\right)^c$, $\sum_1^\infty |Q_j| \le \frac{C}{\lambda}\|f\|_{L^1(R^n)}$, and $|f|_{Q_j} \le C\lambda$, where $(\dots)_{Q_j}$ denotes the mean of the given function over Q_j. It is not necessary that $Q_j \subseteq \overline{\Omega}$ for all j, although since $\operatorname{supp} f \subseteq \overline{\Omega}$ we may assume $Q_j \cap \Omega \ne \varnothing$ for all j. Denote by Q_j^* the dilation of Q_j about its center y_j by a factor of 10.

Now segregate the cubes: write $\mathbb{N} = N_1 \cup N_2$, where $j \in N_1$ if $\operatorname{diam}(Q_j) \le \operatorname{dist}(Q_j, \Omega^c)$, and $j \in N_2$ if $\operatorname{diam}(Q_j) > \operatorname{dist}(Q_j, \Omega^c)$; $j \in N_2$ includes the case $Q_j \not\subseteq \Omega$.

The idea of the proof is as follows. As usual, given a cube Q_j consider only those x not in Q_j^*. When $j = 1$ and $|x - y_j| \le C\delta(y_j)$, $|\nabla_y K(x, y)| \le C|x - y|^{-n-1}$ for $y \in Q_j$, and the proof follows the usual method of showing weak-type bounds for Calderon-Zygmund kernels. In the remaining cases $j = 1$, $|x - y_j| \ge C\delta(y_j)$, and $j = 2$, x sees the mass of $|f|$ at Q_j as being close to $\partial\Omega$. These cases are resolved without the use of cancellation by the estimate $|K(x, y)| \le C|x - y|^{-n}$ and the result of Sjögren [Sj] that the convolution of $|x|^{-n}$ with a measure supported in a "small" set in $\mathbb{R}^n$, like $\partial\Omega$, is in weak $L^1(\mathbb{R}^n)$.

Define g a.e. by letting $g = f$ on $\left(\bigcup_1^\infty Q_j\right)^c$, $g = f_{Q_j}$ on Q_j for $j \in N_1$, and $g = 0$ on Q_j for $j \in N_2$. Define b_j a.e. for $j \in \mathbb{N}$ by $b_j = f - f_{Q_j}$ on Q_j if $j \in N_1$, $b_j = f$ on Q_j if $j \in N_2$, and $b_j = 0$ on Q_j^c. $f = g + \sum_{j=1}^\infty b_j$, and g and b_j are supported in Ω. Clearly,

$$\begin{aligned} &|\{x \in \Omega : |Kf(x)| > \lambda\}| \\ &\quad \le \left|\left\{x \in \Omega : |Kg(x)| > \frac{\lambda}{2}\right\}\right| \\ &\qquad + \sum_{i=1,2} \left|\left\{x \in \Omega \setminus \bigcup_1^\infty Q_j^* : \left|K\left(\sum_{j \in N_i} b_j\right)(x)\right| > \frac{\lambda}{4}\right\}\right| + \left|\bigcup_1^\infty Q_j^*\right|. \end{aligned} \tag{13}$$

We bound the first and last terms of the right-hand side in the usual manner (see [St]). By the lemma, $\|Kg\|_{L^2(R^n)} \le C(d)\|g\|_{L^2(\Omega)}$. In addition, $|g| \le$

$C\lambda$ and $\|g\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$, hence

$$\left|\left\{x \in \Omega: |Kg(x)| > \frac{\lambda}{2}\right\}\right| \leq \frac{4}{\lambda^2}\int |Kg(x)|^2\,dx \leq \frac{C(d)}{\lambda^2}\|g\|^2_{L^2(\Omega)} \leq \frac{C(d)}{\lambda}\|f\|_{L^1(\Omega)}.$$

Also, $|\bigcup_1^\infty Q_j^*| \leq 10^n \sum_1^\infty |Q_j| \leq \frac{C}{\lambda}\|f\|_{L^1(\Omega)}$.

Consider now the summand $i = 1$ in the third term of (13). Define functions χ_j and Λ_j by putting $\chi_j(x) = 1$ if $x \in \Omega\backslash Q_j^*$ and $|x - y_j| \leq 10\delta(y_j)$, and $\chi_j(x) = 0$ otherwise. Let $\Lambda_j(x) = 1$ if $x \in \Omega$ and $|x - y_j| > 10\delta(y_j)$, and $\Lambda_j(x) = 0$ otherwise. Fix j and suppose $\chi_j(x) = 1$ for now. Since b_j has mean value 0 when $j \in N_1$, $Kb_j(x) = \int[K(x, y) - K(x, y_j)]b_j(y)\,dy$. For $y \in Q_j$, $|K(x, y) - K(x, y_j)| \leq \operatorname{diam}(Q_j)\sup_{y'\in Q_j}|\nabla_{y'}K(x, y')|$. Since $y' \in Q_j$ and $j \in N_1$ imply $|x - y'| \leq C\delta(y')$, it follows from (9) that $|\nabla_{y'}K(x, y')| \leq C|x - y'|^{-n-1} \leq C|x - y_j|^{-n-1}$. Hence

$$|Kb_j(x)| \leq C\operatorname{diam}(Q_j)|x - y_j|^{-n-1}\|b_j\|_{L^1(R^n)}$$

when $\chi_j(x) = 1$. Since $\operatorname{diam}(Q_j)\int_{(Q_j^*)^c}|x - y_j|^{-n-1}\,dx \leq C$,

$$\sum_{j\in N_1}\int \chi_j(x)|Kb_j(x)|\,dx \leq C\sum_{j\in N_1}\|b_j\|_{L^1} \leq C\|f\|_{L^1(\Omega)}.$$

By Markov's inequality, $|\{x: \sum_{j\in N_1}\chi_j(x)|Kb_j(x)| > \frac{\lambda}{8}\}| \leq \frac{C}{\lambda}\|f\|_{L^1(\Omega)}$. To finish estimating the size of the $i = 1$ term in (13), it suffices to show that

$$\left|\left\{x: \sum_{j\in N_1}\Lambda_j(x)|Kb_j(x)| > \frac{\lambda}{8}\right\}\right| \leq \frac{C}{\lambda}\|f\|_{L^1(\Omega)}.$$

Inequality (8) implies $|K(x, y)| \leq C|x - y|^{-n}$. When $j \in N_1$, $\Lambda_j(x) = 1$, and $y \in Q_j$, $|x - y|^{-n} \leq C|x - p_j|^{-n}$, where $p_j \in \partial\Omega$ is chosen satisfying $\operatorname{dist}(p_j, Q_j) = \operatorname{dist}(\partial\Omega, Q_j)$ for $j \in \mathbb{N}$ (if $j \in N_2$, p_j is possible in Q_j). Hence

$$\Lambda_j(x)|Kb_j(x)| \leq C\int |x - p_j|^{-n}|b_j(y)|\,dy = C|x - p_j|^{-n}\|b_j\|_{L^1(\Omega)}. \tag{14}$$

Define a measure μ with support in $\partial\Omega$ by

$$\mu = \sum_{j\in N_1}\|b_j\|_{L^1(\Omega)}\delta_{p_j},$$

where δ_{p_j} is the unit point mass at p_j. Let r^{-n} be the function $z \mapsto |z|^n$ on $\mathbb{R}^n$. Then by (14)

$$\sum_{j\in N_1}\Lambda_j(x)|Kb_j(x)| \leq Cr^{-n} * \mu,$$

where $*$ is convolution on $\mathbb{R}^n$. By a theorem of Sjögren (see [Sj]),

$$\left|\left\{x : \sum_{j\in N_1}\Lambda_j(x)|Kb_j(x)| > \frac{\lambda}{8}\right\}\right| \leq |\{x : r^{-n} * \mu(x) > C\lambda\}|$$

$$\leq \frac{C}{\lambda}\operatorname{mass}(\mu) = \frac{C}{\lambda}\sum_{j\in N_1}\|b_j\|_{L^1(\Omega)} \leq \frac{C}{\lambda}\|f\|_{L^1(\Omega)}.$$

(The constant arising from the application of Sjögren's result to $\partial\Omega$ may be taken as uniform over all convex $\Omega \subseteq \mathbb{R}^n$.)

The $i = 2$ summand of the third term of (13) is treated in the same manner as $\sum_{j\in N_1} \Lambda_j(x)|Kb_j(x)|$, because $j \in N_2$, $Q_j \cap \Omega \neq \varnothing$, $x \in \Omega\backslash Q_j^*$, and $y \in Q_j$ imply (with (8)) $|K(x, y)| \leq C|x-y|^{-n} \leq C|x-p_j|^{-n}$. That b_j does not necessarily have vanishing mean value when $j \in N_2$ makes no difference.

3. Proof of Corollary 1

Let $f \in L_s^p(\Omega)$. We first show that $Gf \in L_{s+2}^p(\Omega)$ and that (4) holds for $u = Gf$.

Applying the real interpolation method to (2) and Lemma 1 gives $\|\nabla^2(Gf)\|_{0,p} \leq C(p)\|f\|_{0,p}$, $1 < p \leq 2$. (5) and Young's inequality show that $\|Gf\|_{0,p} \leq C(d)\|f\|_{0,p}$. Similarly, (7) and Young's inequality show that $\|\nabla(Gf)\|_{0,p} \leq Cd\|f\|_{0,p}$. Hence (4) holds for $s = 0$ and $1 < p \leq 2$.

Remark. When Ω is not convex, Corollary 1 may fail for $s = 0$. For example, Jerison and Kenig [JK] found a bounded C^1 domain Ω with $f \in C_0^\infty(\Omega)$, yet $\nabla^2(Gf) \notin L^1(\Omega)$.

We now consider (4) for $s = -1$, $1 < p < \infty$. Suppose $f \in L_{-1}^p(\Omega)$. Note that there exists an $\mathbf{h} \in L^p(\Omega)$ with $f = \operatorname{div}\mathbf{h}$, and $\|\mathbf{h}\|_{L^p(\Omega)} \leq C(d)\|f\|_{-1,p}$. For $s = -1$ we need only show $\|G(\operatorname{div}\mathbf{h})\|_{1,p} \leq C(d, p)\|\mathbf{h}\|_{L^p(\Omega)}$, $1 < p < \infty$. As in the proof of Lemma 2, $G(\operatorname{div}\mathbf{h})(x) = -\int_\Omega \nabla_y G(x, y)\cdot\mathbf{h}(y)\,dy$. It is therefore enough to show that

$$\left\|\int_\Omega \nabla_x\nabla_y G(x, y)h(y)\,dy\right\|_{L^p(\Omega; dx)} + \left\|\int_\Omega \nabla_y G(x, y)h(y)\,dy\right\|_{L^p(\Omega; dx)} \leq C(d, p)\|h\|_{L^p(\Omega)}$$

(where for simplicity of notation we have replaced $\mathbf{h}$ by a scalar-valued h). Real interpolation, (3), and Lemma 2 imply

$$\left\|\int_\Omega \nabla_x\nabla_y G(x, y)h(y)\,dy\right\|_{L^p(\Omega; dx)} \leq C(d, p)\|h\|_{L^p(\Omega)}, \qquad 1 < p \leq 2. \tag{15}$$

Since $\nabla_x\nabla_y G(x, y) = \nabla_y\nabla_x G(y, x)$, duality shows that (15) holds in the range $1 < p < \infty$. That $\|\int_\Omega \nabla_y G(x, y)h(y)\,dy\|_{L^p(\Omega; dx)} \leq C(d, p)\|h\|_{L^p(\Omega; dx)}$ for $1 < p < \infty$ follows from (7) and Young's inequality.

Inequality (4) holds for Gf for the other s and p indicated in Corollary 1 by complex interpolation.

Because G is bounded from $L_{-1}^p(\Omega)$ to $L_1^p(\Omega)$, to see that $Gf \in W_0^{1,p}(\Omega)$, it is enough to check it for f in the dense subclass $L^\infty(\Omega)$. For $f \in L^\infty(\Omega)$, (5) and (7) imply that $Gf(x)$, $\nabla Gf(x)$, and $\delta(x)^{-1}Gf(x)$ are bounded, from which it easily follows that $Gf \in W_0^{1,p}(\Omega)$ for any p, $1 < p < \infty$.

Suppose $u \in W_0^{1,p}(\Omega)$ and $\triangle u = 0$. Given $\psi \in C_0^\infty(\Omega)$, $\int u\triangle\psi\,dx = \int \triangle u\psi\,dx = 0$. The uniqueness asserted in Corollary 1 follows, since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p'}(\Omega)$, $\triangle$ takes $W_0^{1,p'}(\Omega)$ onto $L_{-1}^{p'}(\Omega)$, and $L_{-1}^{p'}(\Omega)$ is the dual of $W_0^{1,p}(\Omega)$ (where $p'^{-1} + p^{-1} = 1$).

Remark. When Ω is only Lipschitz and $s = -1$, the estimate (4) holds for some $p \neq 2$ but not all p, $1 < p < \infty$ (see [JK]). (The case $p = 2$ is just the Dirichlet principle.)

Remark. The theorem and Corollary 1 also hold for Ω bounded, globally Lipschitz, and satisfying a uniform exterior sphere condition. The above proofs are essentially unchanged. Lemma 1 has been shown in this greater generality by Adolfsson (see [Ado2]); Lemma 2 relies only on the Dirichlet principle. Proposition 1 is stated in [GW] in this generality (and the case $n = 2$ still holds); and for the application of Sjögren's result it suffices that Ω be globally Lipschitz. Here, the constants in (2), (3), and (4) will depend on more geometric information about Ω, such as the radius given in the uniform exterior sphere condition.

4. An application to restriction of Sobolev spaces

If Ω is only C^1, it can be shown that Corollary 1 fails for $p = 2$, $s = -\frac{1}{2}$; Jerison and Kenig derive this from the failure of $L^2_{3/2}(\Omega)$ to restrict to $L^2_1(\partial\Omega)$ [JK]. If Ω is convex, we may reverse this logic and show that the restriction $L^2_{3/2}(\Omega) \to L^2_1(\partial\Omega)$ holds. Indeed,

Corollary 2. *If Ω is convex and bounded and $1 < p < \infty$, then the trace of $L^p_{1+1/p}(\Omega)$ to $\partial\Omega$ is the same as the trace of $HL^p_{1+1/p}(\Omega) = \{v : \triangle v = 0, v \in L^p_{1+1/p}(\Omega)\}$.*

Proof. Let $F \in L^p_{1+1/p}(\Omega)$. By Corollary 1, $G(\triangle F) \in L^p_{1+1/p} \cap W^{1,p}_0(\Omega)$. Hence $G(\triangle F)|_{\partial\Omega} = 0$ in the sense of, e.g., $L^p(\partial\Omega)$, and $F|_{\partial\Omega} = (F - G(\triangle F))|_{\partial\Omega}$.

Remark. When $p = 2$ and Ω is Lipschitz, $HL^2_{3/2}(\Omega)|_{\partial\Omega} = L^2_1(\partial\Omega)$ (see [JK]). Therefore if Ω is convex, $L^2_{3/2}(\Omega)|_{\partial\Omega} = L^2_1(\partial\Omega)$.

5. A counterexample

Proposition 2. *There is a bounded convex $\Omega \subseteq \mathbb{R}^n$ and an $f \in C^\infty(\overline{\Omega})$ such that $Gf \notin L^p_{s+2}(\Omega)$ for any p, s with $p > \frac{2}{s+1}$, $-1 < s < 1$, $1 < p < \infty$.*

Without loss of generality take p near $\frac{2}{s+1}$. Pick $r_0 << 1$. Let Ω be the image of $\{z \in \mathbb{C}: |z| < r_0, \operatorname{Im}(z) > 0\}$ under the conformal mapping $\Phi: z \mapsto -z\log z$, where the branch of log is chosen so that $\operatorname{Im}\log z$ lies in $(0, \pi)$ when z is in the upper half plane. Ω lies in the upper half plane and is convex and bounded. Let $\Psi = \Phi^{-1} : \Omega \to \mathbb{C}$ and put $u = \eta \operatorname{Im}\Psi$, where η is the restriction to Ω of a function in $C^\infty_0(\mathbb{C})$ which is 1 in a neighborhood of the origin and is supported very close to the origin (so that u vanishes on $\partial\Omega$). Put $f = \Delta u$. f is supported away from the origin and is $C^\infty(\overline{\Omega})$, and $Gf = u$. For (x, y) near the origin, $|\nabla u(x, y)| = |\Psi'(w)|$, where $w = x + iy$ and

$$\Psi'(w) = \Phi'^{-1}(\Psi(w)) = -\frac{1}{1 + \log\Psi(w)}.$$

As $w \to 0 \in \partial\Omega$, $\Psi' \to 0$. If $\nabla u \in C^\alpha(\overline{\Omega})$ for some $\alpha > 0$, we must therefore have $|\Psi'(w)| = |\nabla u(x, y)| \le C|w|^\alpha$. But a simple calculation shows that $|\Psi'| \ge C|\log|w||^{-1}$ as $w \to 0$, hence $\nabla u \notin C^\alpha(\overline{\Omega})$ for any $\alpha > 0$. This shows that $u \notin L^p_{s+2}(\Omega)$ for $p > \frac{2}{s+1}$, $-1 < s < 1$. For otherwise $\nabla u \in L^p_{s+1}(\Omega) \subseteq C^{s+1-2/p}(\overline{\Omega})$ (see [T, Theorem 4.6.1(e)]).

This example is adapted to $n \ge 3$ by adding extra variables and cutting off.

REFERENCES

[Ada] R. A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

[Ado1] V. Adolfsson, *L^p-integrability of the second order derivatives of Green potentials in convex domains*, Pacific J. Math. (to appear).

[Ado2] ——, *L^2-integrability of second order derivatives for Poisson's equation in nonsmooth domains*, Math. Scand. **70** (1992), 146–160.

[G] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, Boston, MA, 1985.

[GW] M. Grüter and K.-O. Widman, *The Green function for uniformly elliptic equations*, Manuscripta Math. **37** (1982), 202–342.

[JK] D. Jerison and C. E. Kenig, *The functional calculus for the Laplacian on Lipschitz domains*, preprint.

[Sj] P. Sjögren, *Weak L_1 characterizations of Poisson integrals, Green potentials, and H^p spaces*, Trans. Amer. Math. Soc. **233** (1977), 179–196.

[St] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, NJ, 1970.

[T] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland, New York, 1978.

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

Current address: Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario, Canada L8S 4K1

E-mail address: sjfromm@icarus.math.mcmaster.ca