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ON φ-EXTENSIONS OF DEVELOPABLE SPACES

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ABSTRACT. We prove that the φ-extension of Moore spaces is a developable space.

1. INTRODUCTION

In this paper, all the spaces are T_1 and all mappings are continuous and onto. The letter N denotes the positive integers.

For a class $\mathscr{C}$ of topological spaces, Klebanov called a space Y the *φ-extension of* $\mathscr{C}$ if Y is the image of a product space of members of $\mathscr{C}$ under a closed mapping [K_1]. He proved that a φ-extension of metric spaces is metrizable if it is first countable [K_1, Theorem 6; K_2]. From this result, the following question arises: For what classes $\mathscr{C}$ does the φ-extension Y of $\mathscr{C}$ belong to $\mathscr{C}$ if Y is first countable? The author recently obtained positive results for classes of Lašnev spaces, stratifiable spaces, and paracompact σ-spaces but posed the question for the class of developable spaces [M].

In this paper, the author gives a positive answer to the question when $\mathscr{C}$ is a class of Moore spaces, that is, regular developable spaces. A *developable space* is a space X with a sequence $\{\mathscr{U}_n : n \in N\}$ of open covers such that, for each point $p \in X$, $\{St(p, \mathscr{U}_n) : n \in N\}$ is a local base at p in X.

For the sake of brevity, we write $[X_\alpha, A, f, Y]$ for the situation that Y is the image of the product space $\prod\{X_\alpha : \alpha \in A\}$ under a closed mapping f. For an index set A, the product space $\prod\{X_\alpha : \alpha \in A\}$ is briefly denoted by $X(A)$. For a family $\mathscr{H}$ of subsets of a space X and a point $p \in X$, we write

$$C(p, \mathscr{H}) = \bigcap\{H \in \mathscr{H} : p \in H\}.$$

2. MAIN RESULT

To state the main theorem for developable spaces, we need some terminology. Let X be a space. A pair family $\mathscr{P}$ of X is a family of paired subsets $P = (P_1, P_2)$ of X such that $P_1 \subset P_2$, where P_1 is closed and P_2 is open in X. $\mathscr{P}$ is called *discrete in* X if $(\mathscr{P})_1 = \{P_1 : P \in \mathscr{P}\}$ is a discrete family in X, and is called *σ-discrete in* X similarly. $\mathscr{P}$ is called a *pair network for* X if

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$p \in O$ with O open in X and $p \in X$ implies $p \in P_1 \subset P_2 \subset O$ for some $P \in \mathscr{P}$. The next lemma is essentially proved in [B].

Lemma 1. *For a space X, the following are equivalent:*

(1) *X is a developable space.*
(2) *There exists a σ-discrete pair network for X.*
(3) *There exists a σ-locally finite pair network for X.*

For Lemma 2, assume $[X_\alpha, A, f, Y]$. For each $\alpha \in A$ let $\mathscr{F}_\alpha$ be a locally finite family of nonempty closed subsets of X_α, and for each $k \in N$ let Δ_k be the totality of subsets δ of A with $|\delta| = k$. For each $\delta \in \Delta_k$ let

$$Y_\delta = \{y \in Y : H \subset f^{-1}(y) \text{ for some } H \in \mathscr{H}(\delta)\},$$

where

$$\mathscr{H}(\delta) = \prod\{\mathscr{F}_\alpha : \alpha \in \delta\} \times \{X(A - \delta)\}.$$

Lemma 2. *$\bigcup\{Y_\delta : \delta \in \Delta_k\}$ is a discrete closed subset of Y.*

Proof. Without loss of generality, we can assume $Y_\delta \neq \varnothing$ for all $\delta \in \Delta_k$ and $Y_\delta \neq Y_{\delta'}$ if $\delta \neq \delta'$, $\delta, \delta' \in \Delta_k$. First we show that $\delta \cap \delta' \neq \varnothing$ for each $\delta, \delta' \in \Delta_k$. Take points $y \in Y_\delta$, $y' \in Y_{\delta'}$ such that $y \neq y'$. By the definition, there exist $H \in \mathscr{H}(\delta)$, $H' \in \mathscr{H}(\delta')$ such that

$$H \subset f^{-1}(y), \qquad H' \subset f^{-1}(y'),$$

where

$$H = \prod\{F_\alpha : \alpha \in \delta\} \times X(A - \delta),$$
$$H' = \prod\{F'_\alpha : \alpha \in \delta'\} \times Y(A - \delta').$$

If $\delta \cap \delta' \neq \varnothing$ were true, then we would have

$$H \cap H' = \prod\{F_\alpha : \alpha \in \delta\} \times \prod\{F'_\alpha : \alpha \in \delta'\} \times X(A - (\delta \cup \delta')) \neq \varnothing,$$

which is a contradiction to $f^{-1}(y) \cap f^{-1}(y') = \varnothing$. Hence we have $\delta \cap \delta' \neq \varnothing$. Since $|\delta| = k$ for each $\delta \in \Delta_k$, we can show inductively that, for some $t_k \in N$, Δ_k can be written as $\Delta_k = \bigcup\{\Delta(i) : i \leq t_k\}$, where $\{\Delta(i)\}$ has the property that, for each $i \leq t_k$ there exists $\delta_i \subset A$ such that for each δ, $\delta' \in \Delta(i)$ with $\delta \neq \delta'$ we have

$$\delta \cap \delta' = \delta_i \quad \text{and} \quad (\delta - \delta_i) \cap (\delta' - \delta_i) = \varnothing.$$

Since obviously each Y_δ is a discrete closed subset of Y, we can assume $Y_\delta \cap Y_{\delta'} = \varnothing$ if $\delta \neq \delta'$, $\delta, \delta' \in \Delta_k$. Let $i \leq t$ be fixed. For each $y \in Y_\delta$, $\delta \in \Delta(i)$, we take $H(y) \in \mathscr{H}(\delta)$ such that $H(y) \subset f^{-1}(y)$. $H(y)$ is of the form

$$H(y) = \prod\{F_\alpha(y) : \alpha \in \delta\} \times X(A - \delta),$$

where $F_\alpha(y) \in \mathscr{F}_\alpha$, $\alpha \in \delta$. If we set

$$T(y) = \prod\{F_\alpha(y) : \alpha \in \delta_i\} \times X(A - \delta_i),$$

then we easily have

$$Y_\delta \subset f\left(\bigcup\{T(y) : y \in Y_\delta\}\right).$$

From the property of Δ_i and the assumption that $Y_\delta \cap Y_{\delta'} = \varnothing$ for $\delta, \delta' \in \Delta_i$, $\delta \neq \delta'$, it follows that

$$\left\{\bigcup\{T(y) : y \in Y_\delta\} : \delta \in \Delta_i\right\}$$

is hereditarily closure-preserving in $X(A)$. Therefore, $\{Y_\delta : \delta \in \Delta_i\}$ is closure-preserving in Y, and consequently

$$Y(\Delta_i) = \bigcup\{Y_\delta : \delta \in \Delta_i\}$$

is a discrete closed subset of Y. This means that

$$\bigcup\{Y_\delta : \delta \in \Delta_k\} = \bigcup\{Y(\Delta_i) : i \le t\}$$

is a discrete closed subset of Y. This completes the proof.

We call a subset Y of a space X a *σ-discrete closed subset* of X if $Y = \bigcup\{Y_n : n \in N\}$, where each Y_n is a discrete closed subset of X.

Lemma 3. *In the situation $[X_\alpha, A, f, Y]$, we assume that all X_α are Moore spaces and Y is first countable. Then Y is decomposed as $Y = Y_0 \cup Y_1$, where Y_1 is a σ-discrete closed subset of Y and, for each point $y \in Y_0$, $f^{-1}(y)$ is compact in $X(A)$.*

Proof. For each $\alpha \in A$, since X_α is a σ-space in the sense of [O], X_α has a network $\mathscr{F}_\alpha = \bigcup\{\mathscr{F}_{\alpha n} : n \in N\}$, where, for each n, $\mathscr{F}_{\alpha n}$ is a locally finite closed cover of X_α such that $\mathscr{F}_{\alpha n} \subset \mathscr{F}_{\alpha n+1}$ and any finite intersection of members of $\mathscr{F}_{\alpha n}$ belongs to $\mathscr{F}_{\alpha n}$. Let Δ be the totality of finite subsets of A, and for each $\delta \in \Delta$ with $|\delta| = k$ and each $n \in N$ let

$$\mathscr{H}(\delta, n) = \prod\{\mathscr{F}_{\alpha n} : \alpha \in \delta\} \times \{X(A - \delta)\}$$

and define a subset $Y(\delta, n)$ of Y by

$$Y(\delta, n) = \{y \in Y : H \subset f^{-1}(y) \text{ for some } H \in \mathscr{H}(\delta, n)\}.$$

Set

$$Y(k, n) = \bigcup\{Y(\delta, n) : \delta \in \Delta_k\}, \qquad k, n \in N,$$
$$Y_1 = \bigcup\{Y(k, n) : k, n \in N\}, \qquad Y_0 = Y - Y_1,$$

where $\Delta_k = \{\delta \in \Delta : |\delta| = k\}$. By Lemma 2, Y_1 is a σ-discrete closed subset of Y. Therefore, it remains to show that, for each $y \in Y_0$, $f^{-1}(y)$ is compact in $X(A)$. Assume that $y \in Y_0$ and $f^{-1}(y)$ is not compact in $X(A)$. Since a Moore space is subparacompact, there exists $\alpha_0 \in A$ such that

(1) $\qquad p_0(f^{-1}(y))$ is not countably compact in X_{α_0},

where $p_0 \colon X(A) \to X_{\alpha_0}$ is the projection. We settle the following:

(2) $\qquad p_0(f^{-1}(y))$ is Lindelöf.

Since $\mathscr{F}_{\alpha_0}$ is a network for X_{α_0}, it suffices to show that for each n

$$\mathscr{F}_n(y) = \{F \in \mathscr{F}_{\alpha_0 n} : F \cap p_0(f^{-1}(y)) \neq \varnothing\}$$

is finite. Assume that, for some n, $\mathscr{F}_n(y)$ is infinite. Let $\{O_n(y) : n \in N\}$ be a decreasing local base at y in Y. We take $\{F_m : m \in N\} \subset \mathscr{F}_n(y)$, which implies

$$p_0^{-1}(F_m) = H_m \in \mathscr{H}(\{\alpha_0\}, n) \quad \text{for each } m \in N.$$

Observe that, for each $m, n \in N$, $f(H_m) \cap O_n(y)$ is infinite, for otherwise for some $k, m \in N$

$$H_m \cap f^{-1}(O_k(y)) \subset f^{-1}(y),$$

which implies that $H \subset f^{-1}(y)$ for some $H \in \mathscr{H}(\delta, t)$, that is, $y \in Y_1$, a contradiction. From this observation, we can take a sequence $\{p_n : n \in N\} \subset X(A)$ such that $p_n \in f^{-1}(O_n(y)) \cap H_n$ and $f(p_n) \neq f(p_m)$ if $n \neq m$. Then $\{p_n\}$ clusters because of the closedness of f. But this is a contradiction to the discreteness of $\{H_m : m \in N\}$. Hence we can conclude (2). By (1), (2), and the regularity of X_{α_0}, we can easily find an increasing sequence $\{U_k : k \in N\}$ of open subsets of X_{α_0} covering $p_0(f^{-1}(y))$ and satisfying

$$p_0(f^{-1}(y)) \cap (U_{k+1} - \overline{U}_k) \neq \varnothing$$

for each k. Set $V_k = p_0^{-1}(U_k)$, $k \in N$. Since we can show that, for each k, $f^{-1}(O_k(y)) \cap (V_{k+1} - \overline{V}_k)$ is infinite in the same way as above, we can take a sequence $\{z_k : k \in N\} \subset X(A)$ such that

$$z_k \in f^{-1}(O_k(y)) \cap (V_{k+1} - \overline{V}_k), \qquad k \in N.$$

Since f is a closed mapping, $\{z_k\}$ has a cluster point z_0 with $z_0 \in f^{-1}(y)$. But this is a contradiction because $z_0 \in V_k$ for some k. This contradiction means that $f^{-1}(y)$ is compact in $X(A)$, completing the proof.

The main theorem. *A first countable φ-extension of Moore spaces is a developable space.*

Proof. In the situation $[X_\alpha, A, f, Y]$, we assume that all X_α are Moore spaces and Y is a first countable space. By Lemma 3, we consider two cases as follows: (i) Y is a countable union of discrete closed subsets of Y; and (ii) there exists a point $y \in Y$ such that $f^{-1}(y)$ is compact in $X(A)$. In the first case, using the first countability of Y we can easily see that Y has a σ-discrete pair network. Therefore, by Lemma 1 Y is developable. Thus it remains to show that Y is developable under the case (ii). By [K_1, Lemma 1], without loss of generality we can assume that Y is the image of the product space $X \times X(A)$ under a closed mapping f, where X is a Moore space and all X_α are compact metrizable spaces. By Lemma 1, there exists a σ-locally finite pair network $\bigcup\{\mathscr{P}_n : n \in N\}$ for X such that, for each n, $\mathscr{P}_n = \{P_\lambda : \lambda \in \Lambda_n\}$ and $(\mathscr{P}_n)_1 = \{P_{\lambda 1} : \lambda \in \Lambda_n\}$ is a locally finite closed cover of X. Without loss of generality, we can assume that $\bigcup\{(\mathscr{P}_n)_1 : n \in N\}$ is closed under any finite intersection. For each $\lambda \in \Lambda = \bigcup\{\Lambda_n : n \in N\}$, we choose a point $x_\lambda \in P_{\lambda 1}$ arbitrarily. Since

$$Y(\lambda) = f(\{x_\lambda\} \times X(A))$$

is a first countable dyadic space, by [N, Theorem VIII, 11] $Y(\lambda)$ is a compact metrizable space. Let $p_\lambda \colon \{x_\lambda\} \times X(A) \to X(A)$ be the projection, and let $g_\lambda \colon X(A) \to f(\{x_\lambda\} \times X(A)) \subset Y$ be the mapping such that $f/(\{x_\lambda\} \times X(A)) = g_\lambda \cdot p_\lambda$. Since $Y(\lambda)$ is a compact metrizable space, there exists a countable pair network $\{Q_{\lambda n} = (Q_{\lambda n1}, Q_{\lambda n2}) : n \in N\}$ for $Y(\lambda)$. Without loss of generality, we can assume that, for each n, there exists a sequence $\{n_i : i \in N\} \subset N$ such that

$$Q_{\lambda n_1 1} = Q_{\lambda n_2 1} = \cdots = Q_{\lambda n 1},$$
$$(3) \qquad \overline{Q}_{\lambda n_{i+1} 2} \subset Q_{\lambda n_i 2}, \qquad i \in N,$$
$$Q_{\lambda n 1} = \bigcap \{Q_{\lambda n_i 2} : i \in N\}.$$

For each $\lambda \in \Lambda$, $m \in N$, let

$$H(\lambda, m)_1 = P_{\lambda 1} \times g_\lambda^{-1}(Q_{\lambda m 1}), \qquad H(\lambda, m)_2 = P_{\lambda 2} \times g_\lambda^{-1}(Q_{\lambda m 2}).$$

Set

$$\mathscr{H}_1 = \{H(\lambda, m)_1 : \lambda \in \Lambda \text{ and } m \in N\}$$

and

$$\mathscr{H}_2 = \{H(\lambda, m)_2 : \lambda \in \Lambda \text{ and } m \in N\}.$$

We can easily introduce an equivalence relation $\sim$ on $X \times X(A)$ by the following: For each pair of points p, q of $X \times X(A)$, $p \sim q$ if and only if $p \in C(q, \mathscr{H}_2)$. Let $Z[X]$ be the quotient space obtained from $X \times X(A)$ by $\sim$ with the quotient mapping $t\colon X \times X(A) \to Z[X]$. It is easy to check by (3) that

$$(4) \qquad C(p, \mathscr{H}_1) = C(p, \mathscr{H}_2) \quad \text{for each point } p \in X \times X(A).$$

We construct a pair family $\mathscr{R}$ of $Z[X]$ as follows:

$$\mathscr{R} = \bigcup \{\mathscr{R}(n_1, \dots, n_k) : n_1, \dots, n_k \in N,\ k \in N\},$$
$$\mathscr{R}(n_1, \dots, n_k) = \{R(\lambda_1, \dots, \lambda_k; n_1, \dots, n_k) : \lambda_1, \dots, \lambda_k \in \Lambda\},$$
$$n_1, \dots, n_k \in N,\ k \in N,$$

where

$$R(\lambda_1, \dots, \lambda_k; n_1, \dots, n_k)$$
$$= (R_1(\lambda_1, \dots, \lambda_k; n_1, \dots, n_k), R_2(\lambda_1, \dots, \lambda_k : n_1, \dots, n_k))$$

and for each $s = 1, 2$

$$R_s(\lambda_1, \dots, \lambda_k; n_1, \dots, n_k) = t\left(\bigcap \{P_{\lambda_i s} \times g_{\lambda_i}^{-1}(Q_{\lambda_i n_i s}) : i \leq k\}\right),$$
$$\lambda_1, \dots, \lambda_k \in \Lambda,\ n_1, \dots, n_k \in N,\ k \in N.$$

By (4), each $R_2(\lambda_1, \dots, \lambda_k; n_1, \dots, n_k)$ is open in $Z[X]$, i.e., $\mathscr{R}$ is a pair family of $Z[X]$. It is easily seen that, for each $n_1, \dots, n_k \in N$, $k \in N$, $\mathscr{R}(n_1, \dots, n_k)$ is a σ-locally finite pair family in $Z[X]$. Thus $\mathscr{R}$ is so in $Z[X]$. To see that $\mathscr{R}$ is a pair network for $Z[X]$, let $[p] \in O$, where $p = (x, y) \in X \times X(A)$ and O is open in $Z[X]$. Then there exists an open rectangle $U \times V$ of $X \times X(A)$ such that

$$\{x\} \times K_p \subset U \times V \subset t^{-1}(O).$$

Since $\bigcup \{\mathscr{P}_n : n \in N\}$ is a pair network for X, there exists $\lambda_1 \in \Lambda$ such that

$$x \in P_{\lambda_1 1} \subset P_{\lambda_1 2} \subset U.$$

Therefore, for some $m_1 \in N$

$$(5) \qquad C(p, \mathscr{H}_2) \subset H(\lambda_1, m_1)_1 \subset H(\lambda_1, m_1)_2 \subset U \times X(A).$$

By virtue of (3) and (4), there exist $m_2, \ldots, m_k \in N$, $\lambda_2, \ldots, \lambda_k \in \Lambda$ such that

$$
\begin{aligned}
C(p, \mathscr{H}) &\subset \bigcap\{H(\lambda_i, m_i)_1 : i = 1, 2, \ldots, k\} \\
&\subset \bigcap\{H(\lambda_i, m_i)_2 : i = 2, \ldots, k\} \subset X \times V.
\end{aligned} \tag{6}
$$

From both (4) and (6), we have

$$
\begin{aligned}
[p] &\in R_1(\lambda_1, \ldots, \lambda_k : m_1, \ldots, m_k) \\
&\subset R_2(\lambda_1, \ldots, \lambda_k : m_1, \ldots, m_k) \subset O.
\end{aligned}
$$

Hence $\mathscr{R}$ is a pair network for $Z[X]$.

We recall Worrell's method in [W] proving that if Y is the image of a developable space under a closed mapping and if each point inverse of Y has a meta-Lindelöf boundary in the domain, then Y is a developable space. Applying essentially this method, we can easily prove that if Y is a first countable image of a developable space under a closed mapping and if each point of Y except a σ-discrete closed subset Y_1 has a compact point inverse, then Y is a developable space. Let $g: Z[X] \to Y$ be a mapping such that $f = g \cdot t$. Then obviously g is a closed mapping. By Lemma 3 Y is decomposed as $Y = Y_1 \cup Y_0$, where Y_1 is a σ-discrete closed subset of Y and, for each $y \in Y_0$, $f^{-1}(y)$ is compact in $X(A)$. This implies that $g^{-1}(y)$ is compact in $Z[X]$. Thus, by the above we can conclude that Y is a developable space. This completes the proof.

Remark. By the same argument, we can prove that a first countable φ-extension of σ-spaces, that is, regular spaces having a σ-locally finite closed network in the sense of Okuyama [O], has a σ-locally finite closed network.

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