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PROJECTIVELY FLAT SURFACES IN $\mathbb{A}^3$

FABIO PODESTÀ

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ABSTRACT. We consider a nondegenerate immersion $f: M^2 \to \mathbb{A}^3$ of an orientable 2-dimensional manifold M^2 together with the Blaschke connection ∇ induced on M^2; this work is aimed at studying locally convex surfaces whose Blaschke connection is projectively flat, reducing the problem of their classification to a system of PDE's. In particular we can prove the existence of locally convex projectively flat surfaces which are not locally symmetric.

1. PRELIMINARIES AND PROJECTIVE FLATNESS

We consider a nondegenerate immersion $f : M^2 \to \mathbb{A}^3$ of a 2-dimensional orientable surface M^2 into the 3-dimensional affine space $\mathbb{A}^3$; it is well known that there is a transversal vector field ξ, unique up to sign and called *affine normal*, which induces the Blaschke connection ∇ according to the equations

$$D_X(f_* Y) = f_*(\nabla_X Y) + h(X, Y)\xi, \qquad D_X\xi = -f_*(SX)$$

where D denotes the standard flat connection in A^3, and X and Y are vector fields on M^2. We recall here the fundamental structure equations, referring to [1] for a more detailed exposition:

$$\begin{gathered} R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY, \\ (\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z), \\ (\nabla_X S)Y = (\nabla_Y S)X, \\ h(SX, Y) = h(X, SY). \end{gathered}$$

We shall now write down the condition of projective flatness in terms of the affine invariants of M^2. An affinely connected manifold of dimension 2 is projectively flat if and only if the covariant derivative of its Ricci tensor is a totally symmetric (0,3)-tensor. Now the Ricci tensor is given by

$$\operatorname{Ric}(X, Y) = h(X, Y)\operatorname{tr} S - h(SX, Y),$$

hence

$$\begin{aligned} (\nabla_Z \operatorname{Ric})(X, Y) = {} & (\nabla_Z h)(X, Y)\operatorname{tr} S + Z(\operatorname{tr} S)h(X, Y) \\ & - (\nabla_Z h)(SX, Y) - h((\nabla_Z S)X, Y). \end{aligned}$$

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So ∇ is projectively flat if and only if

$$Z(\operatorname{tr} S)h(X, Y) - (\nabla_Z h)(SX, Y) = X(\operatorname{tr} S)h(Z, Y) - (\nabla_X h)(SZ, Y)$$

where we have used Codazzi and Ricci equations. Now, since

$$(\nabla_X h)(Y, Z) = -2h(K_X Y, Z),$$

we get

$$X(\operatorname{tr} S)h(Z, Y) - Z(\operatorname{tr} S)h(X, Y) = 2[h(K_Z SX, Y) - h(K_X SZ, Y)];$$

hence since h is nondegenerate,

$$X(\operatorname{tr} S)Z - Z(\operatorname{tr} S)X = 2[K_Z SX - K_X SZ]$$

or equivalently

$$X(\operatorname{tr} S) = -2\operatorname{tr}(K_X S). \tag{1.1}$$

2. The strongly convex case

We shall now suppose that the quadratic form h is positive definite everywhere. First of all note that affine spheres are projectively flat, as follows immediately from equation (1.1), when $S = \lambda\,\mathrm{Id}$. So we shall suppose that M^2 is not an affine sphere.

Since S is symmetric with respect to h and h is positive definite, we can diagonalize S; so we can find local vector fields e_1, e_2 such that

$$h(e_i, e_j) = \delta_{ij} \quad \text{and} \quad S(e_i) = \lambda_i e_i$$

for some C^∞ functions $\lambda_1 \neq \lambda_2$. Now we shall write

$$\nabla_{e_1} e_2 = re_1 + se_2, \qquad \nabla_{e_2} e_1 = pe_1 + qe_2$$

for some C^∞-functions p, q, r, s.

Now if we write down the Codazzi equation for S, we find immediately that

$$q(\lambda_1 - \lambda_2) = e_1(\lambda_2), \qquad r(\lambda_1 - \lambda_2) = -e_2(\lambda_1).$$

Moreover, from the flatness condition (1.1) we get

$$e_1(\lambda_1 + \lambda_2)e_2 - e_2(\lambda_1 + \lambda_2)e_1 = 2(\lambda_1 - \lambda_2)K_{e_1} e_2.$$

Now since $\{e_1, e_2\}$ is an orthonormal basis for h, we have

$$h(K_{e_1} e_2, e_2) = h(\nabla_{e_1} e_2, e_2), \qquad h(K_{e_1} e_2, e_1) = h(\nabla_{e_2} e_1, e_1)$$

so that

$$s = \frac{e_1(\lambda_1 + \lambda_2)}{2(\lambda_1 - \lambda_2)}, \qquad p = -\frac{e_2(\lambda_1 + \lambda_2)}{2(\lambda_1 - \lambda_2)}.$$

Summing up we have

$$\begin{aligned} \nabla_{e_1} e_2 &= -\frac{e_2(\lambda_1)}{\lambda_1 - \lambda_2} e_1 + \frac{e_1(\lambda_1 + \lambda_2)}{2(\lambda_1 - \lambda_2)} e_2, \\ \nabla_{e_2} e_1 &= -\frac{e_2(\lambda_1 + \lambda_2)}{2(\lambda_1 - \lambda_2)} e_1 + \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} e_2. \end{aligned} \tag{2.1}$$

So if we put $\phi = \frac{1}{2}\log|\lambda_1 - \lambda_2|$, we get

$$[e_1, e_2] = -e_2(\phi)e_1 + e_1(\phi)e_2,$$

and we can choose local coordinates (x, y) so that

$$\frac{\partial}{\partial x} = e^{-\phi}e_1, \qquad \frac{\partial}{\partial y} = e^{-\phi}e_2.$$

Thus we can write

$$\nabla_{\partial/\partial x}\frac{\partial}{\partial y} = -\frac{1}{\lambda_1 - \lambda_2}\frac{\partial\lambda_1}{\partial y}\frac{\partial}{\partial x} + \frac{1}{\lambda_1 - \lambda_2}\frac{\partial\lambda_2}{\partial x}\frac{\partial}{\partial y}. \tag{2.2}$$

Now we put

$$\nabla_{e_1}e_1 = \alpha e_1 + \beta e_2, \qquad \nabla_{e_2}e_2 = \gamma e_1 + \delta e_2,$$

and using the apolarity condition we get

$$\alpha = h(\nabla_{e_1}e_1, e_1) = -h(\nabla_{e_1}e_2, e_2),$$
$$\delta = h(\nabla_{e_2}e_2, e_2) = -h(\nabla_{e_2}e_1, e_1).$$

Using the first Codazzi equation we get

$$\beta = h(\nabla_{e_1}e_1, e_2) = -h(\nabla_{e_1}e_2, e_1) + 2h(\nabla_{e_2}e_1, e_1)$$

and

$$\gamma = h(\nabla_{e_2}e_2, e_1) = 2h(\nabla_{e_1}e_2, e_2) - h(\nabla_{e_2}e_1, e_2).$$

Summing up we get

$$\nabla_{e_1}e_1 = -\frac{e_1(\lambda_1 + \lambda_2)}{2(\lambda_1 - \lambda_2)}e_1 - \frac{e_2(\lambda_2)}{\lambda_1 - \lambda_2}e_2, \tag{2.3}$$

or equivalently

$$\nabla_{\partial/\partial x}\frac{\partial}{\partial x} = -\frac{1}{\lambda_1 - \lambda_2}\frac{\partial\lambda_1}{\partial x}\frac{\partial}{\partial x} - \frac{1}{\lambda_1 - \lambda_2}\frac{\partial\lambda_2}{\partial y}\frac{\partial}{\partial y}, \tag{2.4}$$

and

$$\nabla_{e_2}e_2 = \frac{e_1(\lambda_1)}{\lambda_1 - \lambda_2}e_1 + \frac{e_2(\lambda_1 + \lambda_2)}{2(\lambda_1 - \lambda_2)}e_2, \tag{2.5}$$

or equivalently

$$\nabla_{\partial/\partial y}\frac{\partial}{\partial y} = -\nabla_{\partial/\partial x}\frac{\partial}{\partial x}. \tag{2.6}$$

Now the last integrability conditions are given by the Gauss equation, which can be expressed as

$$R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\frac{\partial}{\partial x} = -\frac{\lambda_2}{|\lambda_1 - \lambda_2|}\frac{\partial}{\partial y}$$

and

$$R\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)\frac{\partial}{\partial y} = \frac{\lambda_1}{|\lambda_1 - \lambda_2|}\frac{\partial}{\partial x}.$$

These two equations can be rewritten as a system of partial differential equations in the unknowns λ_1, λ_2; supposing $\lambda_1 > \lambda_2$, they are as follows:

$$\frac{\partial\lambda_1}{\partial x}\frac{\partial\lambda_2}{\partial y} = \frac{\partial\lambda_1}{\partial y}\frac{\partial\lambda_2}{\partial x}, \tag{2.7}$$

$$\frac{\partial^2\lambda_2}{\partial x^2}+\frac{\partial^2\lambda_2}{\partial y^2}+\frac{2}{\lambda_1-\lambda_2}\left[\left(\frac{\partial\lambda_2}{\partial x}\right)^2+\left(\frac{\partial\lambda_2}{\partial y}\right)^2\right]=-\lambda_2,\tag{2.8}$$

$$\frac{\partial^2\lambda_1}{\partial x^2}+\frac{\partial^2\lambda_1}{\partial y^2}-\frac{2}{\lambda_1-\lambda_2}\left[\left(\frac{\partial\lambda_1}{\partial x}\right)^2+\left(\frac{\partial\lambda_1}{\partial y}\right)^2\right]=\lambda_1.\tag{2.9}$$

Remark. A direct computation shows that at each point of the open subset U where $\lambda_1 \neq \lambda_2$, we have

$$\mathrm{Tr}_h(\nabla S)=0,\tag{2.10}$$

and since (2.10) holds also on the interior part of the complement of U, we get that (2.10) holds everywhere by continuity. Hence if M^2 is supposed to be an ovaloid, by the same argument used in [2, Theorem 1] we conclude that M^2 is a quadric.

3. Nonlocally symmetric examples

The main problem in classifying at least locally projectively flat affine surfaces consists in integrating equations (2.7)–(2.9).

In this section we shall indicate how to get locally convex projectively flat surfaces which are not locally symmetric (w.r.t. the Blaschke connection). In order to do this, we shall first write down the condition $\nabla R = 0$ using our notation. We shall consider the open subset U of M^2 where $\lambda_1 \neq \lambda_2$.

First of all we note that the condition of projective flatness implies (see [3]) that the covariant derivative of the curvature tensor $(\nabla_W R)(X, Y)Z$ is symmetric in the arguments Z, W. Moreover, using equations (2.1), (2.3), and (2.5) we get

$$(\nabla_{e_1}R)(e_1, e_2)e_2=-\frac{1}{\lambda_1-\lambda_2}(e_1(\lambda_1\lambda_2)e_1+e_2(\lambda_1\lambda_2)e_2),$$

$$(\nabla_{e_1}R)(e_1, e_2)e_1=\frac{1}{\lambda_1-\lambda_2}(e_2(\lambda_1\lambda_2)e_1-e_1(\lambda_1\lambda_2)e_2),$$

and since $(\nabla_{e_1}R)(e_1, e_2)e_1=-(\nabla_{e_2}R)(e_1, e_2)e_2$, we conclude that

$$\nabla R=0 \Longleftrightarrow \lambda_1\lambda_2=\text{constant}.$$

Moreover, by taking traces of the equations above in order to get the expression for the covariant derivative $\nabla\,\mathrm{Ric}$, we have the following.

Proposition 3.1. *For a projectively flat locally convex affine surface the following conditions are equivalent*:

(1) $\nabla\,\mathrm{Ric}=0$.
(2) $\nabla R=0$.
(3) $\det S=\text{const}$.

From this last proposition it follows that every such surface with $\det S=0$ is locally symmetric; if this is the case, we look for solutions (λ_1, λ_2) with

$$\lambda_2=0, \qquad \lambda_1>0.$$

Hence equations (2.7)–(2.9) are reduced to

$$\frac{\partial^2 \log\lambda}{\partial x^2} - \left(\frac{\partial \log\lambda}{\partial x}\right)^2 + \frac{\partial^2 \log\lambda}{\partial y^2} - \left(\frac{\partial \log\lambda}{\partial y}\right)^2 = 1 \tag{3.1}$$

where we have put $\lambda = \lambda_1$. We note, moreover, that under the substitution $\mu = -1/\lambda$, equation (4.1) reduces to

$$\frac{\partial^2 \mu}{\partial x^2} + \frac{\partial^2 \mu}{\partial y^2} = -\mu. \tag{3.2}$$

Hence the system of PDE's governing the immersion f is

$$f_{xx} = \frac{1}{\mu}\frac{\partial \mu}{\partial x} f_x - \mu\xi, \quad f_{yy} = -\frac{1}{\mu}\frac{\partial \mu}{\partial x} f_x - \mu\xi, \quad f_{xy} = \frac{1}{\mu}\frac{\partial \mu}{\partial y} f_x,$$
$$\xi_x = \frac{1}{\mu} f_x, \qquad \xi_y = 0.$$

In order to give just an example of a surface arising this way, we note that $\mu = -\cos y$, with $y \in (-\pi/2, \pi/2)$, is a solution of (3.2). So if we fix initial conditions $f(0, 0) = 0 \in \mathbb{A}^3$, $f_x(0, 0) = {}^t(1, 0, 0)$, $f_y(0, 0) = {}^t(0, 1, 0)$, and $\xi(0, 0) = {}^t(0, 0, 1)$, we find the immersion

$$f(x, y) = {}^t(\operatorname{sen} x \cos y,\ y,\ 1 - \cos x \cos y).$$

We note that after a minor coordinate change the surface is part of $x_1^2 + x_2^2 = (\cos x_3)^2$, which is a rotation surface generated by the cosine. If we use the solution $\mu = -\cos x$, with $x \in (-\pi/2, \pi/2)$, we get a translation surface which can be represented as

$$f(x, y) = {}^t(x, y, \tfrac{1}{2}y^2 + g(x))$$

where $g(x) = \frac{1}{8}(2x^2 + 2x \operatorname{sen}(2x) + \cos(2x) - 1)$; it would be nice to have a complete classification of all locally convex translation surfaces which are locally symmetric.

If $\det S = k \neq 0$, then it is easy to see that the integrability conditions are reduced to

$$\frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 \lambda}{\partial^2 y} - \frac{2\lambda}{\lambda^2 - k}\left[\left(\frac{\partial \lambda}{\partial x}\right)^2 + \left(\frac{\partial \lambda}{\partial y}\right)^2\right] = \lambda \tag{3.3}$$

where $\lambda = \lambda_1$, $\lambda_2 = k/\lambda_1$, and $\lambda^2 > k$.

In order to get examples which are not locally symmetric, we have to find solutions of the system (2.7)–(2.9) for which the product $\lambda_1\lambda_2$ is not constant.

Proposition 3.2. *There exist locally convex projectively flat surfaces which are not locally symmetric*

Proof. Just to simplify notation, we consider solutions of the system (2.7)–(2.9) with $\partial\lambda_i/\partial x = 0$ so that each λ_i is a function of the variable y only. Then the integrability conditions are reduced to the system (the prime symbol $'$ denotes differentiation with respect to the variable y)

$$\begin{cases} \lambda_1'' - \dfrac{2}{\lambda_1 - \lambda_2}(\lambda_1')^2 = \lambda_1 \\ \lambda_2'' + \dfrac{2}{\lambda_1 - \lambda_2}(\lambda_2')^2 = -\lambda_2 \end{cases}$$

which can be rewritten as a system of partial differential equations of the first order by putting $f_1 = \lambda_1$, $f_2 = \lambda_1'$, $f_3 = \lambda_2$, and $f_4 = \lambda_2'$:

$$\begin{cases} f_1' = f_2, \\ f_2' = \dfrac{2}{f_1 - f_3}(f_2)^2 + f_1, \\ f_3' = f_4, \\ f_4' = -\dfrac{2}{f_1 - f_3}(f_4)^2 - f_3. \end{cases}$$

If we now choose initial values $f_1(0) = 1$, $f_2(0) = a$, $f_3(0) = 0$, $f_4(0) = 1$ with any real number a, by the Cauchy theorem we get locally a solution (λ_1, λ_2), $\lambda_1 > \lambda_2$, whose product is not constant since $(\lambda_1\lambda_2)'(0) = f_1(0)f_4(0) = 1 \neq 0$. □

Remark. When the second fundamental form h is not definite but the shape operator is still diagonalizable, we can choose local vector fields e_1, e_2 with $h(e_1, e_1) = 1$, $h(e_2, e_2) = -1$, $h(e_1, e_2) = 0$, $S(e_1) = \lambda_1 e_1$, $S(e_2) = \lambda_2 e_2$ and again we can find local coordinates (x, y) with $\partial/\partial x = e^{-\phi}e_1$, $\partial/\partial y = e^{-\phi}e_2$ using the same notation as in §2. Then the integrability conditions are only slightly modified, namely,

$$\frac{\partial\lambda_1}{\partial x}\frac{\partial\lambda_2}{\partial y} = \frac{\partial\lambda_1}{\partial y}\frac{\partial\lambda_2}{\partial x},$$

$$\frac{\partial^2\lambda_2}{\partial x^2} - \frac{\partial^2\lambda_2}{\partial y^2} + \frac{2}{\lambda_1 - \lambda_2}\left[\left(\frac{\partial\lambda_2}{\partial x}\right)^2 - \left(\frac{\partial\lambda_2}{\partial y}\right)^2\right] = -\lambda_2,$$

$$\frac{\partial^2\lambda_1}{\partial y^2} - \frac{\partial^2\lambda_1}{\partial x^2} + \frac{2}{\lambda_1 - \lambda_2}\left[\left(\frac{\partial\lambda_1}{\partial x}\right)^2 - \left(\frac{\partial\lambda_1}{\partial y}\right)^2\right] = \lambda_1,$$

and Propositions 3.1 and 3.2 are still valid in the hyperbolic case.

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Dipartimento di Matematica "U. Dini", Viale Morgagni 67/A, I-50134 Firenze, Italy