

A CYCLIC MONOTONICALLY NORMAL SPACE WHICH IS NOT K_0

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ABSTRACT. We construct a space as in the title, thus answering a variety of questions.

We construct a space T as in the title, answering questions raised in [MRRC, vMR, MR, M, H, vD].

For T to be monotonically normal [B, Z], points must be closed, and, for every point u and open neighborhood U of u , there must be an open neighborhood $H(u, U)$ such that

- (1) for $u \notin V$ and $v \notin U$, $H(u, U) \cap H(v, V) = \emptyset$, and
- (2) for $U \subset V$, $H(u, U) \subset H(u, V)$.

Such a function H is called a monotone normality operator for T .

If G is a monotone normality operator for T and $n \geq 3$, we say that a sequence $\{y^i | i \leq n\}$ from T is an n -cycle of G if $y^0 = y^n$ and

$$\bigcap_{i < n} G(y^i, T - \{y^{i-1}\}) \neq \emptyset;$$

T would be called *acyclically* monotonically normal if it had an operator without any cycles. We show that if G is a monotone normality operator for T , then G has a cycle of length 3. That is, we find a set $\{y^i | i < 3\}$ of three points such that

$$\bigcap_{i < 3} G(y^i, T - \{y^{i-1}\}) \neq \emptyset.$$

(Here and throughout the paper we add i 's (and j 's) modulo 3. Thus $y^3 = y^0$.) Hence T gives a strong negative answer to the following frequently asked question, first asked in [MRRC]: is every monotonically normal space acyclically monotonically normal? Equivalent questions are known [MRRC, M, MR].

For each $n \in \omega$, Kuratowski [K] defined a class K_n of spaces. Both K_0 spaces and monotonically normal spaces are K_1 [vD], while acyclically monotonically normal spaces are K_0 [MR]; T is K_1 but not K_0 . Thus T gives a negative answer to the following, again frequently asked [H, MRRC, M, MR,

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vMR], questions, first asked by van Douwen in his thesis [vD]: is every K_1 space K_0 (partially answered by van Mill [vM]) and is every monotonically normal space K_0 ?

Also T is first-countable, 0-dimensional, hereditarily separable, hereditarily Lindelöf, and hereditarily paracompact [BR], quite a nice space in many ways.

Define $T = 3^{\leq \omega}$; let \leq be the usual partial order on T defined by $t \leq s$ in T if s extends t . Then $\langle T, \leq \rangle$ is a tree of height $\omega+1$ which has 3^n as its n th level. Let $X = 3^{<\omega}$ and $Y = 3^\omega$ (the top level of the tree); $T = X \cup Y$. Each $x \in X$ has exactly three immediate successors $\{x_0, x_1, x_2\}$. For $x < y \in Y$ in the tree T , define

$$B_x(y) = \{y\} \cup \{z \in X \mid x < z < y\} \\ \cup \{t \in T \mid \exists z \in X \text{ with } x < z < y \text{ and } i < 3 \text{ with } z_i < y \text{ and } z_{i+1} \leq t\}.$$

(Recall that we add i 's modulo 3.) Our topology on T is the one induced by using $\{\{x\} \mid x \in X\} \cup \{B_x(y) \mid x < y \in Y\}$ as a basis.

We make several observations.

I. *This is indeed a topology.*

Proof. If $t \in B_x(y) \cap Y$ and z is maximal in X for $z < y$ and $z < t$, then $B_z(t) \subset B_x(y)$. For $y \in Y$, $\{B_x(y) \mid x < y\}$ is a decreasing local basis at y . \square

II. *Points are closed.*

Proof. Suppose $t \neq y$ in T . We need a neighborhood of y missing t . We can assume $y \in Y$. If $t \in X$, $t \in 3^n$ for some $n \in \omega$ and $t \notin B_x(y)$, where x is the term of 3^n with $x < y$. If $t \in Y$, there is a maximal $x \in X$ with $x < t$ and $x < y$; $t \notin B_x(y)$. \square

III. *T is monotonically normal.*

Proof. Define $H(x, U) = \{x\}$ if $x \in X$ and $H(y, U) = B_x(y)$ if $y \in Y$ and x is minimal for $B_x(y) \subset U$.

Since the other conditions necessary for H to be a monotonic normality operator are trivially satisfied, we only need check (1). Suppose $u \in U$ which is open, $v \in V$ which is open, $u \notin V$, and $v \notin U$. We can assume both u and v are in Y .

If t is maximal in X for $t < u$ and $t < v$, then one of $\{t_0, t_1, t_2\}$ is $< u$ and another is $< v$. We can assume that $t_i < u$ and $t_{i+1} < v$. Since $v \notin U$, $H(u, U) = B_x(u)$ for some $x \geq t$. But $B_x(u) \cap B_z(v) = \emptyset$ for all $z < v$. Thus (1) holds and T is monotonically normal. \square

IV. *T is not acyclically monotonically normal.*

Comment. This follows from observation V, but I give a proof, for it is basic to T . I first observed that a triple branching Souslin tree, similarly topologized, has this property. I sent this information to Moody who independently discovered a "real" example.

Proof. Suppose G is a monotone normality operator for T . Let r be the unique first term of T (its root). For each $y \in Y$ there is an $f(y) < y$ in T such that $B_{f(y)}(y) \subset G(y, B_r(y))$.

Claim. There are $x \in X$ and $\{y^1, y^2, y^3\} \subset Y$ such that, for $i < 3$, $f(y^i) < x < x_i < y^i$.

Suppose our claim is true. For each i observe that, since $y^{i-1} \notin B_r(y^i)$, $G(y^i, B_r(y^i)) \subset G(y^i, T - \{y^{i-1}\})$. And $x \in B_{f(y^i)}(y^i) \subset G(y^i, B_r(y^i))$. So

$$x \in \bigcap_{i < 3} G(y^i, T - \{y^{i-1}\}).$$

Thus G has a 3-cycle as desired.

To prove our claim, suppose that for every $x \in X$ there is an $i(x) < 3$ such that for all $y \in Y$ with $x_{i(x)} < y$, $f(y) \geq x$.

For each $n \in \omega$ we then choose $x^n \in X$ by induction, taking $x^0 = r$ and $x^{n+1} = x^n_{i(x^n)}$. There is a unique $y \in Y$ which extends x^n for all $n \in \omega$. But there is $n \in \omega$ with $f(y) < x^n$. Since $x^{n+1} = x^n_{i(x^n)} < y$ and $f(y) < x^n$, we have a contradiction to the definition of $i(x^n)$. Our claim thus holds, and T is not acyclically monotonically normal. \square

V. T is not a K_0 space.

In order for a space T with topology τ to be K_0 , it is necessary [K] that, for every $Y \subset T$ having the subspace topology ρ , there be a function $k: \rho \rightarrow \tau$ such that, for all U and V in ρ ,

- (1) $k(\emptyset) = \emptyset$,
- (2) $k(U) \cap Y = U$, and
- (3) $k(U \cap V) = k(U) \cap k(V)$.

Assume such a k for our space T and its subspace Y .

(For T to be K_1 for every Y , there is a k satisfying (2) such that $k(U) \cap k(V) = \emptyset$ if $U \cap V = \emptyset$.)

For $p < x \in X$ and $i < 3$ define

$$U_{pxi} = \{u \in X \mid p < u \leq x_i\} \\ \cup \{t \in T \mid \exists u \in X \text{ and } j < 3 \text{ such that } p < u \leq x, u_j \leq x_i, \text{ and } u_{j+1} \leq t\}.$$

Recall that r is the root of T . Define

$$U_{xi} = U_{rxi} \cup \{t \in T \mid x_i < t\}.$$

Our proof that there can be no k as assumed has several steps.

(a) There is $p \in X$ such that, for all $x > p$ in X and $i < 3$, $U_{pxi} \subset k(U_{xi} \cap Y)$.

Proof of (a). Suppose there is no such p . Then by induction, for each $n \in \omega$, we can choose $p^n \in X$ as follows. Let $p^0 = r$. Assume that $p^{2n} = p$ has been chosen. By our supposition, there are $x > p$ and $i < 3$ such that $U_{pxi} \not\subset k(U_{xi} \cap Y)$. Choose $p^{2n+1} = x$ and $p^{2n+2} = x_i$.

There is a unique $y \in Y$ which extends p^n for all $n \in \omega$. By property (2) of k , there is some $z \in X$ such that $B_z(y) \subset k(B_r(y) \cap Y)$ and some $n \in \omega$ such that $z < p^{2n} = p$. Then if $x = p^{2n+1}$ and $x_i = p^{2n+2}$, $U_{pxi} \not\subset k(U_{xi} \cap Y)$. But $U_{pxi} \subset B_p(y) \subset B_z(y)$ and $B_r(y) \subset U_{xi}$, so $U_{pxi} \subset k(U_{xi} \cap Y)$, which is a contradiction. \square

We keep p satisfying (a) fixed for the rest of the proof.

(b) If $p < t \in X$ and $j < 3$, then $U_{ptj} \subset k(U_{rtj} \cap Y)$.

Proof of (b). To see this, simply observe that if $x = t_j$, $\bigcap_{i < 3} U_{pxi} = U_{ptj}$ and $\bigcap_{i < 3} U_{xi} = U_{rtj}$. Thus from (a) and property (3) we get (b). \square

(c) S_n holds for all $n \in \omega$.

Definition and proof for (c). If $n \in \omega$, $p < x \in X$, and $i < 3$, let $V_{nxi} = \{v \in X \mid x_i \leq v \text{ and } n \text{ is the number of terms of } X \text{ between } x_i \text{ and } v\}$. Let S_n be the statement $V_{nxi} \subset k(U_{rx i} \cap Y)$ for all x and i .

We prove the S_n 's by induction.

S_0 holds, since $V_{0xi} = \{x_i\}$ and $x_i \in U_{pxi} \subset k(U_{rx i} \cap Y)$ by (b).

Suppose S_n holds and $v \in V_{(n+1)xi}$ for some x and i . Since S_n holds, $v \in V_{nxi} \subset k(U_{rx i} \cap Y)$, where $(x_i)_j \leq v$. Observe that by the definition of $U_{rx i(y-1)}$, $v \in U_{rx i(j-1)} \subset k(U_{rx i(j-1)} \cap Y)$ by (b). Thus by property (3), $v \in k(U_{rx i} \cap Y) = k(U_{rx i} \cap Y)$. Thus S_{n+1} holds, and we have proved (c). \square

An immediate consequence of (c) is

(d) If $p < x \in X$ and $i < 3$, then $\{v \in X \mid x_i < v\} \subset k(U_{rx i} \cap Y)$.

(e) Completion of the proof that k fails to have the desired properties.

Fix x and i as in (d), and choose $y \in Y$ extending x_i . By property (2) there is $z \in X$ with $x_i < z < y$ such that $B_z(y) \subset k(B_{x_i}(y) \cap Y)$. Choose $v \in X$ with $z < v < y$; then $v \in B_z(y) \subset k(B_{x_i}(y) \cap Y)$. We have $B_{x_i}(y) \cap U_{rx i} = \emptyset$, so $k(B_{x_i}(y) \cap U_{rx i} \cap Y) = \emptyset$ by property (1). But $v \in k(U_{rx i} \cap Y)$ by (d), and $v \in k(B_{x_i}(y) \cap Y)$. Hence $k(U_{rx i} \cap Y) \cap k(B_{x_i}(y) \cap Y) \neq \emptyset$, contradicting property (3).

Comments. 1. T is strongly monotonically normal. (See [H].) That is, T has a monotonic normality operator G such that $t \in G(y, U)$ implies $G(t, U) \subset G(y, U)$. If $y \in X$, define $G(y, U) = \{y\}$. If $y \in Y$ and x is minimal for $B_x(y) \subset U$, then define

$$G(y, U) \cup \{t \in T \mid \exists z \in X \text{ with } x < z < y \\ \text{and } i < 3 \text{ with } z_i \leq t \text{ and } \{w \in T \mid z_i \leq w\} \subset U\}.$$

2. Questions of A. V. Arhangel'skii. Does there exist a compact K_0 -space which is not K_1 ? Or a compact monotonically normal space which is not acyclically monotonically normal?

REFERENCES

- [B] C. Borges, *On stratifiable spaces*, Pacific J. Math. **17** (1966), 1–16.
- [BR] Z. Balogh and M. E. Rudin, *Monotonic normality*, Topology Appl. **47** (1992), 115–127.
- [vD] E. van Douwen, *Simultaneous extensions of continuous functions*, Thesis, Vrije Universiteit, Amsterdam, 1975.
- [H] R. Heath, *Stronger forms of monotonic normality* (to appear).
- [K] K. Kuratowski, *Topology*, vol. I, Academic Press, New York, 1966.
- [M] P. Moody, *Acyclic monotonically normal spaces*, Thesis, Mathematical Institute, Oxford, 1988.
- [MR] P. Moody and W. Roscoe, *A cyclic monotone normality*, Topology Appl. **47** (1992), 53–67.

- [vM] J. van Mill, *The reduced measure algebra and a K_1 -space which is not K_0* , Topology Appl. **13** (1982), 123–132.
- [vMR] J. van Mill and G. M. Reed, *Open problems in topology*, North-Holland, Amsterdam, 1990.
- [MRRC] P. Moody, M. Reed, W. Roscoe, and P. Collins, *A lattice of conditions on topological spaces*. II, §3, Fund. Math. (to appear).
- [Z] P. Zenor, *Monotonically normal spaces*, Notices Amer. Math. Soc. **17** (1970), 1034.

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