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LYAPUNOV CHARACTERISTIC EXPONENTS ARE NONNEGATIVE

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ABSTRACT. We prove that, for an arbitrary rational map f on the Riemann sphere and an arbitrary probability invariant measure on the Julia set, Lyapunov characteristic exponents are nonnegative a.e. In particular $\log|f'|$ is integrable. An analogous theorem is proved for smooth maps of an interval with all critical points being nonflat.

This allows us to fill a gap in the proof of Denker and Urbański's theorem that there exists a probability conformal measure on the Julia set with exponent equal to the supremum of the Hausdorff dimensions of probability invariant measures with positive entropy.

0. INTRODUCTION

Our main aim is to prove the following

Theorem A. *Let f be a rational mapping of the Riemann sphere $\overline{\mathbb{C}}$ and μ an arbitrary probability f-invariant measure on the Julia set $J = J(f)$. Then for μ-almost every point $x \in J$*

$$\chi(x) = \lim_{n\to\infty} \log|(f^n)'(x)| \geq 0. \tag{1}$$

In particular, the function $\log|f'|$ is μ-integrable.

This easily yields

Corollary A. *For μ-almost every $x \in J$, $\limsup_{n\to\infty} |(f^n)'(x)| \geq 1$.*

The same methods (Koebe-like distortion lemma + the nonexistence of homtervals,which replaces Montels's Theorem) give

Theorem B. *For every $f\colon I \to I$ a smooth map of the interval with a finite number of critical points all being nonflat and μ an arbitrary probability f-invariant measure on $\mathscr{J}^{\mathbb{R}}$, which denotes the complement of the domain of attraction to sinks and neutral points here, for μ-almost every point $x \in J^{\mathbb{R}}$ the formula* (1) *holds.*

One could ask whether, given f, $\chi(x) > 0$ for every μ and μ-a.e. x. Sometimes it is true; for example, it is obvious if f is expanding on J (i.e.,

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if $|(f^n)'| > 1$ for some n). It is also true for subexpanding f, namely, such that the ω-limit set $\omega(\mathrm{Crit} \cap J)$ is disjoint with Crit, where $\mathrm{Crit} = \mathrm{Crit}(f)$ denotes the set of critical points for f, and there are no neutral rational points in J. This uses the recent Mañé's result that f is expanding on $\omega(\mathrm{Crit} \cap J)$ [M].

However, $\chi = 0$ for μ supported by a "neutral set". Then we call μ a neutral measure. This happens, for example, for neutral points in $J(f)$ with Dirac measures on them, for harmonic measure on the boundary of a Siegel disc, or even in the case $J(f) = \overline{\mathbb{C}}$ for the invariant probability measure on the unit circle for some maps of the form

$$z \to \lambda z \frac{z-a}{1-\bar{a}x}\left(\frac{z-\gamma a}{1-\gamma\bar{a}z}\right)^{-1} \qquad \text{for } |\lambda| = |\gamma| = 1\,, \quad \gamma \approx 1\,;$$

see [H, GPS].

Corollary A is useful already in a weaker form, namely, that there exists at least one x in every closed forward invariant subset of J such that $\limsup_{n\to\infty} |(f^n)'(x)| \geq 1$. This fact allows one to prove that the supremum of Hausdorff dimensions of probability f-invariant positive entropy measures on $J(f)$ is equal to the smallest exponent δ for which a Sullivan's conformal measure exists (i.e., a probability measure with Jacobian $|f'|^\delta$). This equality was almost proved in [DU], and the above-mentioned version of Corollary A was just the missing link.

We end this introduction with a sketch of the proof of Theorem A in the case there exists only one critical c in $J(f)$ because in this case the main idea is very transparent. We consider the Riemann sphere with the standard conformal metric; the distances and $|f'(x)|$ are considered with respect to it; for any $x \in J$, $r > 0$, $B(x, r)$ denotes the ball with the origin at x and radius r in this metric.

Step 1. There exists a constant $C > 0$ such that, for every $n \geq 0$, $\mathrm{dist}(f^n(c), c) > \exp -nC$. Otherwise f^n maps $B = B(c, 2\exp -nC)$ into itself and so the family $f^{kn}|_B$, $k = 1, \dots$, is normal. Hence, $c \notin J$, a contradiction.

Step 2. Case (1). Assume that for a constant $C > 0$, $x \in J$, and every n large enough we have $\mathrm{dist}(f^n(x), c) > \exp -nC$. Then

$$\limsup_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)| \geq -C\,. \tag{2}$$

Otherwise the diameters of $f^n(B(x, \varepsilon))$ for ε small enough shrink at least like $|(f^n)'(x)|\tau^n$ for τ arbitrarily close to 1 due to the fact that the contraction is stronger than approaching by $f^n(x)$ the critical point (this is a standard Pesin's theory consideration). Again $f^n|_{B(x,\varepsilon)}$ would be a normal family. Thus if $\chi(x)$ exists, we get $\chi(x) \geq -C$.

Case (2). If there exist n arbitrarily large with $\mathrm{dist}(f^n(x), c) \leq \frac{1}{2}\exp -nC$, then for C as in Step 1, there is a univalent branch f_ν^{-n} on $B = B(c, \exp -nC)$ such that $f_\nu^{-n}(B) \ni x$ and with the use of Koebe's distortion lemma we obtain

$$|(f^n)'(x)| \geq \mathrm{Const}\,\mathrm{diam}(B) = \mathrm{Const}\exp -nC\,,$$

since, roughly speaking, $f_\nu^{-n}(B)$ cannot be larger than $\overline{\mathbb{C}}$. Hence, we again obtain (2), and if $\chi(x)$ exists we get $\chi(x) \geq -C$.

The idea is that the "bad" case where $f^n(x)$ is close to c happens to be "good" because the critical point is the safest from being contaminated by critical values for iterates of f. The model situation is the subexpanding case where the critical values cannot approach c at all.

Step 3. $\chi(x) \geq -C$ for a constant $C > 0$ and μ-almost every x implies by the Birkhoff Ergodic Theorem the integrability of $\log|f'|$ which implies $\chi \geq 0$. This is standard.

1. Finiteness features

Here f is a rational mapping of the Riemann sphere; we keep the notation from §0.

Lemma 1. *Let $c \in \mathrm{Crit} \cap J$. There exists a constant $C_1 > 0$ such that, for every ε, $n > 0$, if $f^n(B(c, \varepsilon)) \cap B(c, \varepsilon) \neq \varnothing$ then $n \geq C_1 \log \frac{1}{\varepsilon}$.*

Proof. We may assume ε is small, if it is large C_1 takes care of it. Then $\operatorname{diam} f(B(c, 2\varepsilon)) \leq 10\varepsilon^2$ and $\operatorname{diam} f^n(B(c, 2\varepsilon)) \leq 10\varepsilon^2 L^{n-1}$ where $L = \sup_{x \in \overline{\mathbb{C}}} |f'(x)|$. If $10\varepsilon^2 L^{n-1} < \varepsilon$, then as $f^n(B(c, 2\varepsilon))$ intersects $B(c, \varepsilon)$ we obtain $f^n(B(c, 2\varepsilon)) \subset B(c, 2\varepsilon)$, so c is in a domain of normality for the iterates f^n, which contradicts $c \in J$. Thus, $10\varepsilon^2 L^{n-1} \geq \varepsilon$; hence, $n \geq (\log \frac{1}{10\varepsilon}) / \log L$, which proves our lemma.

Lemma 2 (finiteness lemma). *There exists a constant $M > 0$ such that for every $c \in \mathrm{Crit} \cap J$, for every $n \geq 0$ there are at most M critical values for f^n in $B = B(c, \exp -n)$.*

Proof. Suppose that for $c_1 \in \mathrm{Crit}$, $k > l \geq 0$, we have $f^k(c_1)$, $f^l(c_1) \in B$. Then by Lemma 1 $k - l \geq C_1 \log(\frac{1}{\exp -n}) = C_1 n$. So for $M = (C_1^{-1} + 1)\sharp \mathrm{Crit}$ Lemma 2 is satisfied.

In the sequel we shall refer to the following Mañé's lemma [M]:

Lemma 3 (Mañé's lemma). *Given $\varepsilon > 0$, $0 < k < 1$, $c > 0$, and $N > 0$ there exists $\delta > 0$ such that, if for a disc $B = B(x, \delta)$ we have $\mathrm{dist}(B, p) > c$ for every neutral rational or attracting periodic point p and for some $n > 0$ and a component V of $f^{-n}(B)$ there are not more than N critical points of f^n in V, then $\operatorname{diam} W \leq \varepsilon$ for every component W of $f^{-n}(B(x, k\delta)) \cap V$.*

In fact, the inductive consideration proving the next lemma also proves Mañé's lemma for $N > 0$ if one has it already for $N = 0$.

Lemma 4 (bounded distortion lemma). *There exists $C_2 > 0$, such that for every $c \in \mathrm{Crit} \cap J$ and $n \geq 0$ there exists r, $\frac{1}{2} < r < 1$, such that for every j, $0 \leq j \leq n$, every component D_r of $f^{-n}(B_r)$, where we write $B_\tau = B(c, \tau \exp -n)$ for every τ, $0 < \tau \leq 1$, the following holds:*

for every two points $y \in \partial D_r$, $z \in D_r \cap f^{-n}(B_{1/2})$

$$\mathrm{dist}(f^j(y), f^j(z)) \geq C_2 \operatorname{diam} f^j(D_r). \tag{3}$$

Proof. Let D be an arbitrary component of $f^{-n}(B_{3/4})$. We need to consider only large n. So due to Lemma 2 and Mañé's lemma we may assume that the diameters of all $f^j(D)$, $j = 0, \ldots, n$, are small. Due to Lemma 2 there exist $\frac{1}{2} \leq r_1 < r_2 \leq \frac{3}{4}$, $r_2 - r_1 \geq \frac{1}{4M}$ such that there are no critical values for f^n

in $\operatorname{cl} B_{r_2} \setminus B_{r_1}$. Consider an arbitrary component D_{r_2} of $f^{-n}(B_{r_2})$ in D. For $r_1 \le \tau \le r_2$ denote $f^{-n}(B_\tau) \cap D_{r_2}$ by D_τ.

Observe that all sets $f^j(D_{r_2})$ are simply connected, $f^j(D_{r_2} \setminus \operatorname{cl} D_{r_1})$ are topological annuli, and the degree of f^n on $D_{r_2} \setminus \operatorname{cl} D_{r_1}$ is at most $d^{M\sharp\mathrm{Crit}}$ where d majorizes multiplicities of f at critical points.

We prove it by induction. For $j = n$ everything has been assumed. Assume the above assertion for $j \le n$. Then $f^{j-1}(D_{r_2})$ contains at most one critical point for f because it has small diameter. This critical point is in fact in $f^{j-1}(D_{r_1})$. So $f^{j-1}(D_{r_2})$ is simply connected, $f^{j-1}(D_{r_2} \setminus \operatorname{cl} D_{r_1})$ is a topological annulus, and the degree of f on it is the multiplicity of it at the critical point. Formally from Lemma 2 it follows that we meet critical point with $f^j(D_{r_2})$ for at most $M\sharp\mathrm{Crit}$ number of j's. (We multiply M by $\sharp\mathrm{Crit}$ as the orbit of one critical point may hit another critical point so they give the same critical value for f^n in B_{r_2}. In fact, from the proof of Lemma 2 it follows that we can omit the factor $\sharp\mathrm{Crit}$ as we shall in the sequel.)

Let

$$r'' = r_2 - \tfrac{1}{3}(r_2 - r_1), \qquad r' = r_1 + \tfrac{1}{3}(r_2 - r_1).$$

The distortion (i.e., supremum of the ratios of the absolute values of derivatives over all pairs of points) of each branch of $f^{-n(n-j)}$ on, say, each half of the annulus $B_{r''} \setminus B_{r'}$ (for example, $0 \le \operatorname{Arg}(z-c) \le \pi$ or $\pi \le \operatorname{Arg}(z-c) \le 2\pi$) to $f^j(D_{r''} \setminus D_{r'})$ is bounded by a constant depending only on M (use Koebe's distortion lemma). Finally we have degree of f^{n-j} on $f^j(D_{r''} \setminus D_{r'})$ bounded by d^M. This yields the lemma for $r = r''$.

Lemma 5. *There exist κ, $C_3 > 0$ such that for every n, x if $f^n(x) \in B_{1/2}$ then there exist j, Δ satisfying $0 \le j < j + \Delta \le n$, $\Delta \ge n\kappa$ such that*

$$|(f^\Delta)'(f^j(x))| \ge C_3 \exp -2n.$$

Proof. Take $D_{1/2} \ni x$ as in Lemma 4. We have

$$\operatorname{diam} f^n(D_{1/2}) / \operatorname{diam} D_{1/2} \ge (\exp -n)/(2 \operatorname{diam} \overline{\mathbb{C}}).$$

Hence, for some $T \le M + 1$

$$\prod_{t=1}^{T} \left(\frac{\operatorname{diam} f^{j_t}(D_{1/2})}{\operatorname{diam} f^{j_{t-1}+1}(D_{1/2})} \right) \ge \operatorname{Const} \exp -n \tag{4}$$

where $j_0 = 0$, $j_T = n$, and, for $t = 1, \ldots, T-1$, $j_t < j_{t+1}$ are consecutive integers for which $f^{j_t}(D_r)$ intersects Crit (r from Lemma 4).

Denote $\Delta_t = j_t - j_{t-1} - 1$. By Lemma 4 and Koebe's distortion lemma f^{Δ_t} for every t has distortion on $f^{j_{t-1}+1}(D_{1/2})$ bounded by a constant depending only on C_2. Hence from (4) for a constant C_4 depending only on C_2 we get

$$\prod_{t=1}^{T} |(f^{\Delta_t})'(f^{j_{t-1}+1}(x))| > C_4^T \exp -n. \tag{5}$$

If each $A_t = |(f^{\Delta_t})'(f^{j_{t-1}+1}(x))|$ for $\Delta_t > \kappa n$ were smaller than $\exp -2n$ (κ will be specified below), then

$$\prod A_t \le \prod_{\Delta_t > \kappa n} A_t \prod_{\Delta_t \le \kappa n} A_t \le (\exp -2n) L^{\kappa n T}.$$

We used the fact that there exists $\Delta_t > \kappa n$, which holds if $\kappa n T + T < n$. If we assume additionally $1 > \kappa T \log L$, then we get $\prod A_t < \exp -n(1+\tau)$ for some $\tau > 0$, which contradicts (5) for n large.

2. $\chi(x) \geq 0$ FOR A RATIONAL MAPPING OF $\overline{\mathbb{C}}$

In this section we shall prove Theorem A. We start with a lemma standard in Pesin theory [P]:

Lemma 6. *For every* $x \in \overline{\mathbb{C}}$ *if*

$$\limsup_{n\to\infty} \frac{1}{n}(\log|(f^n)'(x)| - \log \operatorname{dist}(f^n(x), \mathrm{Crit})) < 0 \tag{6}$$

then there exists $\tau > 0$ *such that*

$$\lim_{n\to\infty} \operatorname{diam} f^n(B(x, \tau)) = 0; \tag{7}$$

more precisely,

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\operatorname{diam} f^n(B(x, \tau))}{|(f^n)'(x)|} = 0. \tag{8}$$

Proof. Denote $A = \limsup_{n\to\infty} \log(|(f^n)'(x)| - \log \operatorname{dist}(f^n(x), \mathrm{Crit}))$. By (6) for every $\lambda > 1$ satisfying $\log \lambda < |A|$ and for every $\varepsilon > 0$ there exists $\tau > 0$ such that for every $n \geq 0$

$$\frac{\tau|(f^n)'(x)|\lambda^n}{\operatorname{dist}(f^n(x), \mathrm{Crit})} < \varepsilon.$$

Assume that $\varepsilon < 1$. We have for every $y, z \in B_n := B(f^n(x), \tau|(f^n)'(x)|\lambda^n)$ the inequality

$$\left| \frac{|f'(y)|}{|f'(z)|} - 1 \right| \leq \frac{\sup_{a,b\in B_n} |f'(a)| - |f'(b)|}{\inf_{a\in B_n} |f'(a)|} := R_n.$$

We have $\operatorname{diam} B_n$ converging to 0 exponentially fast. So in the case B_n is far from Crit, the ratio R_n is exponentially small. In the opposite case choose a closest critical point to B_n. Denote the multiplicity of f at this critical point by d. We have

$$R_n \leq \mathrm{Const} \frac{\operatorname{dist}(B_n, \mathrm{Crit})^{d-2} \operatorname{diam} B_n}{\operatorname{dist}(B_n, \mathrm{Crit})^{d-1}} \leq \mathrm{Const}\, \varepsilon.$$

For ε small enough this implies $f(B_n) \subset B_{n+1}$, so $f^n(B(x, \tau)) \subset B_n$ and diameters converge to 0, which proves (7).

Taking λ arbitrarily close to 1 one obtains the inequality $\leq$ in (8). The other side of the inequality follows from the existence of a universal bound for the distortions of all f^n on $B(x, \tau)$.

Corollary. *If* $x \in J$ *and* $\operatorname{dist}(f^n(x), \mathrm{Crit}) \geq \exp -n\delta$ *for all* n *large enough then* $\chi(x) \geq -\delta$.

Proof. If the assertion were false then by Lemma 6 we would find τ for which (7) holds. Hence, the family f^n would be normal on $B(x, \tau)$, which is contrary to the assumption $x \in J$.

Proof of Theorem A. First we prove that there exists a constant Const such that for μ-almost every $x \in J$ we have $\chi(x) \geq \text{Const} > -\infty$. Let E be the set in J on which χ exists (we allow the value $-\infty$). By the Birkhoff Ergodic Theorem $\mu(E) = 1$. Fix an arbitrary $\vartheta > 0$. For every $C > 0$ let $E(C)$ denote the set of $x \in E$ such that for every $n \geq 0$

$$\log|(f^n)'(x)| < n(\chi(x) + \vartheta) + C \quad \text{if } \chi(x) > -\infty,$$

(9) or

$$\log|(f^n)'(x)| < -n\vartheta^{-1} + C \quad \text{if } \chi(x) = -\infty.$$

Of course $\bigcup_{C>0} E(C) = E$. Fix an arbitrary C with $\mu(E(C)) > 0$. Consider the function Φ equal to 1 on $E(C)$ and 0 outside. Again by the Birkhoff Ergodic Theorem for almost every $x \in E(C)$ there exists a nonzero limit $\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \Phi(f^j(x))$; hence, for all j large enough (depending on x) we have

$$n_{j+1}(x) - n_j(x) < \frac{\kappa}{2} n_j \tag{10}$$

where $n_j(x)$ denote all consecutive integers for which $f^{n_j(x)}(x) \in E(C)$ and κ is from Lemma 5. We now have two possibilities:

1. Suppose that for every n large enough $\text{dist}(f^n(x), \text{Crit}) > \frac{1}{2}\exp -n$. In this case $\chi(x) \geq -2$ by the Corollary.

2. Suppose now that there exist n arbitrarily large and $c \in \text{Crit} \cap J$ such that $f^n(x) \in B(c, \frac{1}{2}\exp -n)$. Then by Lemma 5 $(f^\Delta)'(f^j(x)) \geq C_3 \exp -2n$. As $\Delta \geq \kappa n$ we find by (10) $t \geq 0$, $t < \frac{\kappa}{2}n$ such that $f^{j+t}(x) \in E(C)$. We have

$$|(f^{\Delta-t})'(f^{j+t}(x))| \cdot L^t \geq |(f^\Delta)'(f^j(x))|$$

(recall that $L = \sup|f'|$). Hence,

$$|(f^{\Delta-t})'(f^{j+t}(x))| \geq L^{-\kappa n/2} C_3 \exp -2n$$

and in view of (9)

$$\exp((\Delta - t)(\chi(f^{j+t}(x)) + \vartheta) + C) \geq L^{-\kappa n/2} C_3 \exp -2n$$

or

$$\exp(-(\Delta - t)\vartheta^{-1} + C) \geq L^{-\kappa n/2} C_3 \exp -2n$$

in the case $\chi(f^{j+t}(x)) = -\infty$

Hence, as χ is f-invariant and as we may restrict our consideration only to $\chi + \vartheta$ negative we get

$$n\frac{\kappa}{2}(\chi(x) + \vartheta) + C \geq -2n + \log C_3 - n\frac{\kappa}{2}\log L$$

or a corresponding inequality for $\chi = -\infty$.

Taking ϑ arbitrarily small (positive) and n large we see that the case $\chi = -\infty$ leads to a contradiction, so we arrive at

$$\chi(x) \geq -4/\kappa - \log L.$$

In particular, we have proved that $\int \log|f'(x)|\,d\mu(x) = \int \log\chi(x)\,d\mu(x) > -\infty$; i.e., $\log|f'|$ is μ-integrable.

Now we shall prove that, for almost every x, $\chi(x) \geq 0$.

For an arbitrary $c \in \mathrm{Crit} \cap J$ and every $\delta > 0$ denote $B_n = B(c, \exp -n\delta)$. Then for n_0 large enough, denoting the multiplicity of f at c by d, we have

$$\begin{aligned} -\infty < \int_{B_{n_0}} \log|f'|\,d\mu &\leq \mathrm{Const} \sum_{n \geq n_0} \int_{B_n \setminus B_{n+1}} \log \mathrm{dist}(x, c)^{d-1}\,d\mu(x) \\ &\leq \mathrm{Const}(d-1) \sum_{n \geq n_0} (-n\delta)\mu(B_n \setminus B_{n+1}) \\ &= -\mathrm{Const}(d-1)\delta \sum_{n \geq n_0} \mu(B_n). \end{aligned}$$

So the series $\sum \mu(B_n)$ is convergent and, by the f-invariance of μ, $\sum \mu(f^{-n}(B_n))$ is convergent. So by the Borel-Cantelli lemma, for almost every x, $f^n(x) \notin B(c, \exp -n\delta)$ for n sufficiently large. Doing the same for every $c \in \mathrm{Crit} \cap J$ we obtain $\mathrm{dist}(f^n(x), \mathrm{Crit}) \geq \exp -n\delta$ for every $\delta > 0$ and n sufficiently large. So by the Corollary we obtain $\chi(x) \geq 0$.

Proof of Corollary A. Suppose the assertion of Corollary A is false. Then there exist $E \subset J$, an integer n_0, and $0 < C < 1$ such that $\mu(E) > 0$ (in fact, $\mu(E)$ arbitrarily close to 1), and for every $x \in E$, $n \geq n_0$ one has $|(f^n)'(x)| < C < 1$.

Given x denote $J(n) = \{j : 0 \leq j \leq n,\ f^j(x) \in E\}$. By the Birkhoff Ergodic Theorem for almost every $x \in E$ there exists $\alpha = a(x) > 0$ such that $\#(J(n)) \geq \alpha n$ for all n large enough. So taking into account only each n_0th j from $J(n)$ and next neglecting the last one and indexing them $j_1, j_2, \ldots, j_m$ we obtain under the convention $j_0 = 0$, $j_{m+1} = n$

$$|(f^n)'(x)| = \prod_{s=0}^{m} |(f^{j_{s+1}-j_s}(f^{j_s}(x)))' < C^{\alpha n/n_0 - 1}.$$

This implies $\chi(x) \leq \frac{\alpha}{n_0} \log C < 0$, which contradicts Theorem A.

3. The interval case

The proof of Theorem B is basically the same as in the complex case, so we list only the places where there are differences.

1. The proof of the analogon of Lemma 1 (frequency lemma) should be changed because it is not sufficient to arrive at the situation $f^n(B(c, \varepsilon)) \subset B(c, \varepsilon)$ contradictory in the complex case. In the interval case this happens even for ε arbitrarily small, for unimodal ∞-renormalizable maps [CE]. The modified proof is as follows:

Fix $n > 0$ and suppose that there exists a critical point c not in a basin of attraction to a periodic sink or a neutral point such that

$$f^n([c-\varepsilon, c+\varepsilon]) \cap [c-\varepsilon, c+\varepsilon] \neq \varnothing. \tag{11}$$

Fix an arbitary ε_0 for which there exists a critical point c_0 so that (11) holds and

$$\varepsilon_0 \leq 2\inf\{\varepsilon : (11) \text{ holds for } \varepsilon \text{ and some } c\}. \tag{12}$$

If $n < \log \frac{1}{10\varepsilon_0} / \log L$ then for every $j = 1, \dots, n$ under the notation $J_j := f^j[(c_0 - 2\varepsilon_0, c_0 + 2\varepsilon_0])$ we have

$$|J_j| \le 5\varepsilon_0^2 L^n < \varepsilon_0/2 .$$

So there is no critical point in J_j for every $j = 1, \dots, n$. Otherwise, as in (11) $f^n(J_j) \cap J_j \neq \varnothing$, so the assumption (12) would not be satisfied.

We conclude that $f^n(J_0)$ lands in J_0 entirely right or left of c_0 and f^n on $[c_0 - 2\varepsilon_0, c]$ and on $[c, c_0 + 2\varepsilon_0]$ is monotone. This implies the existence of a sink of period n or $2n$ in J_0 which attracts c_0, a contradiction. We conclude that $n \ge \log \frac{1}{20\varepsilon} / \log L$ if (11) is satisfied.

2. The analogon of Lemma 3 (Mañé's lemma) holds even with $k = 1$. However, in the presence of infinitely many sinks or neutral points one needs to fix a closed set $A \subset J^{\mathbb{R}}$ not containing sinks and neutral points and instead of assuming distance of $B(x, d)$ from them greater than c one assumes $\operatorname{dist}(B(x, d), A) < d$.

We prove this for $N = 0$ (then the case $N > 0$ is very easy). If a sequence of intervals (I_n) satisfies the properties $\operatorname{diam} I_n \ge \text{Const} > 0$, $\operatorname{diam} f^n(I_n) \to 0$, and $\operatorname{dist}(f^n(I_n), A) \to 0$, then all the intervals from a subsequence I_{n_j} of (I_n) contain a nondegenerate interval J which is a homterval; i.e., $f^n(J)$ do not contain critical points for all $n = 0, 1, \dots$ and $\operatorname{diam} f^{n_j}(J) \to 0$. So by [MMS] or [BL] $f^n(J)$ converge to a sink or a neutral point, contrary to the convergence of a subsequence to A.

3. In the proofs of the analogons of Lemma 4 and Lemma 5 we use Koebe's distortion lemma in the interval version [S, MS]. Denote the interval being the component of $f^{-(n-j)}(B_{r_2})$ containing $f^j(D_{r_2})$ by D^j, $J = 0, \dots, n$, (r_2 from the proof of Lemma 4) and the analogous sets for r_1 by D'^j. We apply Koebe's distortion lemma to each $f^{-\Delta_t}: D^{j_{t+1}} \to D^{j_t+1}$ (notation Δ_t from the proof of Lemma 5). This is allowed due to the fact that the multiplicity of the covering by the family of (D^j), $j = j_t + 1, \dots, j_{t+1}$, is bounded by a universal constant (finiteness lemma).

Finally for each t we have for each component A of $f^{-1}(D^{j_t+1} \setminus D'^{j_t+1})$ in D^{j_t}

$$\frac{\operatorname{diam} A}{\operatorname{diam} D^{j_t}} \ge \text{Const} \frac{\operatorname{diam} f(A)}{\operatorname{diam} D^{j_t+1}},$$

where Const depends only on the nonflatness at the critical point involved (see Figure 1).

4. In the proof of the analogon of the Corollary from §2 the argument of normality should be replaced by the existence of a homterval J of $\operatorname{diam} f^n(J) \to 0$ so by [BL] $f^n(J)$ converges to a sink or a neutral point (as at point 2).

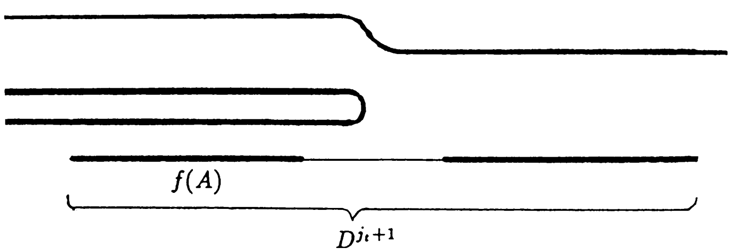

FIGURE 1

ADDED IN PROOF

Theorem B follows easily from the "specification property", more precisely from Corollary 5 of A. M. Blokh, *Decomposition of dynamical systems on an interval*, Russian Math. Surveys **38** (1983), no. 5, 133–134. I owe this observation to F. Hofbauer.

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