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RINGS WITH ANNIHILATOR CHAIN CONDITIONS AND RIGHT DISTRIBUTIVE RINGS

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ABSTRACT. We prove that if a right distributive ring R, which has at least one completely prime ideal contained in the Jacobson radical, satisfies either a.c.c or d.c.c. on principal right annihilators, then the prime radical of R is the right singular ideal of R and is completely prime and nilpotent. These results generalize a theorem by Posner for right chain rings.

INTRODUCTION

The following question occurred in a paper by Posner [9]: Do there exist prime ideals in a right chain ring which are not completely prime? Several authors have approached this problem independently from various points of view (see [1]); however, the question remains open (see [3, 10]), and it is natural to ask for additional conditions which imply that a prime ideal in a right chain ring is completely prime.

In the first part of [9, Theorem 2] it is claimed that if a right chain ring R has either a.c.c. or d.c.c. on right annihilators, then the prime radical $P(R)$ of R is the set of nilpotent elements. This fact implies that the prime radical of R is completely prime. There is a gap in the proof and the chain conditions are needed for right ideals rather than right annihilator ideals, when R is not prime. In fact, it is not proved that the annihilator chain conditions are inherited by $R/P(R)$; however, the result holds and the original motivation of this paper was to find a proof for it.

We say that a ring R is a right distributive ring, or right D-ring for short, if its lattice of right ideals is distributive. It is well known that the class of commutative D-domains coincides with the class of Prüfer domains. The study of noncommutative right D-rings was mainly promoted by a paper of Stephenson [11]. The class of right chain rings (see [1] and the literature quoted therein) is an interesting class of examples. Brungs [2] proved that right D-domains are locally right chain rings. Recently two papers by Mazurek and Puczyłowski [8]

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and Mazurek [7] showed that some features of right chain rings can be carried over to right D-rings.

The purpose of this note is to prove the following:

Theorem 8. *Let R be a right D-ring, which has at least one completely prime ideal contained in the Jacobson radical and satisfies either a.c.c. or d.c.c. on principal right annihilators. Then the prime radical of R equals the right singular ideal of R and is completely prime and nilpotent.*

We say that the ring R satisfies condition (C) if the following holds:

(C) There exists a completely prime ideal Q of R contained in the Jacobson radical J of R.

This condition first appeared in [11, Proposition 2.1(ii)]. Later it was used in [8], and in [7], where it was shown that the condition is of great interest. Let us point out that it is automatically satisfied for a right chain ring R; therefore, our result gives an extension of Posner's assertion.

Throughout this paper, every ring R has a unit element. By $J = J(R)$ we denote the *Jacobson radical*, $P = P(R)$ the *prime radical*, and $A = A(R)$ the *generalized nil radical* of R. Further, we write $r(a) = \{x \mid ax = 0\}$ the principal right annihilator of the element a in R. The notations $\subset$ and $\supset$ will mean strict inclusions. Ideals are assumed to be two-sided unless otherwise stated.

Proof of the theorem

Let R be any ring. By $Z = Z_r(R) = \{x \in R \mid r(x)$ is an essential right ideal of $R\}$ we denote the right singular ideal of R (see [5, pp. 30–36]).

An ideal I of a ring R is said to be right *T-nilpotent* if for every sequence $(x_i)_{i\in\mathbb{N}}$ of elements of I there exists an n such that $x_n x_{n-1} \cdots x_2 x_1 = 0$. We begin with the following:

Lemma 1. *Suppose that R satisfies a.c.c. on principal right annihilators. Then Z is right T-nilpotent, in particular, $Z \subseteq P(R)$.*

Proof. Suppose $(x_i)_{i\in\mathbb{N}}$ is a sequence of elements in Z such that $x_n \cdots x_2 x_1 \neq 0$ for all $n \in \mathbb{N}$. Since $r(x_1) \subseteq r(x_2 x_1) \subseteq \cdots$ is an ascending chain of principal right annihilators, there exists m with $r(x_m b) = r(b)$, $b = x_{m-1} \cdots x_1$. Now x_m is in Z and $b \neq 0$, so $r(x_m) \cap bR \neq 0$ and there exists $y \in R$ with $by \neq 0$, $x_m by = 0$, which is a contradiction. The proof is complete since every ideal which is right T-nilpotent is contained in the prime radical (see [4, Proposition 2.3]). $\square$

We recall the following results [11, Proposition 2.1(ii); 7, Lemma 3.1(ii), Corollary 3.3, and Theorem 3.4] for a right distributive ring R.

Lemma 2. *Let R be a right D-ring and Q a completely prime ideal contained in J.*

(i) *For every right ideal I of R we have $I \subseteq Q$ or $Q \subseteq I$.*
(ii) *For any $a, b \in R$ we have: The elements a, b are comparable, that is, $aR \subseteq bR$ or $bR \subset aR$ or otherwise $aQ = bQ$ holds.*
(iii) *The prime radical P of R is a prime ideal.*
(iv) *There is no two-sided ideal I of R with $P \subset I \subset A$.*

Note that from Lemma 2(i) condition (C) is satisfied in a right D-ring if and only if the generalized nil radical A of R is completely prime. This was already remarked in [8, p. 469]. Obviously this is automatically true provided R is a right chain ring (see [1]).

Now we can prove the following:

Proposition 3. *Let R be a right D-ring which satisfies condition* (C). *Then R is right nonsingular if and only if it is a domain.*

Proof. Assume that $Z = 0$ and let Q be a completely prime ideal of R contained in J. If Q equals zero we are done. So we may assume $Q \neq 0$. Take any nonzero elements $a, b \in R$. By Lemma 2(ii) we have the alternatives $aR \subseteq bR$, $bR \subseteq aR$, or $aQ = bQ$. If $aQ = 0$ holds, then $r(a) \supseteq Q$ and, by Lemma 2(i), $r(a)$ is an essential right ideal of R. Hence $a \in Z = 0$. Therefore we have $aR \cap bR \neq 0$ and so the right Goldie dimension of R is one. Thus $Z = 0$ is a completely prime ideal of R by [7, Proposition 1.2(i)]. Consequently, R is a domain. The converse is obvious. □

Mazurek pointed out to us the following lemma, which was proved by Tuganbaev in a more general setting [12, Lemma 8]. For the sake of completeness, we include an adaption of Tuganbaev's proof to our case.

Lemma 4. *Let R be a right D-ring. Then R/Z has no nonzero nilpotent elements.*

Proof. Assume that $a \notin Z$ and $a^2 \in Z$. Then $H = r(a^2)$ is an essential right ideal of R and $L = r(a)$ is not essential. So there exists a nonzero right ideal of B of R with $L \cap B = 0$ and so $L \cap (H \cap B) = 0$. By [11, Corollary 1(i)$'$ of Proposition 1.1] we have $\mathrm{Hom}_R(H \cap B, L) = 0$, and since $a(H \cap B) \subseteq L$, we get $a(H \cap B) = 0$. Therefore $H \cap B \subseteq r(a) = L$, hence $H \cap B = 0$, a contradiction. □

Now we prove some lemmas, which are necessary for the d.c.c. case.

Lemma 5. *Let R be a right D-ring which satisfies condition* (C). *If I is an ideal of R with $A \not\subseteq I$, then we have $I \subseteq P(R)$.*

Proof. By Lemma 2(i), $I \subset A$. Hence, if $A = P$, we are done; therefore, we may assume $P \subset A$. Suppose there exists $a \in I$ with $a \notin P$, and take any element $b \in P$. Then one of the following three contradictions will follow: (i) $a \in bR \subseteq P$, or (ii) $aA = bA \subseteq P$, which contradicts the primeness of P, or (iii) $b \in aR \subseteq I$. The last possibility would imply $P \subseteq I \subset A$ and so $I = P$. Thus $I \subseteq P$. □

Lemma 6. *Let R be a right D-ring and Q a completely prime ideal contained in J. Then $Q^2 = \{ab \mid a, b \in Q\}$.*

Proof. By induction, it is enough to prove for $x = a_1b_1 + a_2b_2 \in Q^2$ with a_i, $b_i \in Q$ for $i = 1, 2$ that there exist $a, b \in Q$ with $x = ab$. By Lemma 2(ii) we have either $a_1 = a_2y$ resp. $a_2 = a_1y$, for some $y \in R$ or $a_1Q = a_2Q$. In the second case $a_1b_1 = a_2b'$ follows for some $b' \in Q$. The rest is obvious. □

Lemma 7. *Let R be a right D-ring and Q a completely prime ideal of R contained in J. Further assume R satisfies d.c.c. on principal right annihilators. Then we have*

(i) $Z \subseteq Q$;
(ii) *If $Q = Q^2 \neq 0$, then $Z \subset Q$.*

Proof. (i) If $Q = 0$, then R is a domain and $Z = 0$. So we may assume $Q \neq 0$. Suppose $Q \subset Z$ and take any $a \in Z$, $a \notin Q$, and $0 \neq b \in Q$. By [7, Lemma 3.1(i)] we have $Q = aQ$. Hence there exists $c \in Q$ such that $b = ac$. So $r(c) \subseteq r(b)$ and $r(a) \cap cR \neq 0$. Thus we find $x \in R$ with $cx \neq 0$ and $acx = 0$, which implies $r(c) \subset r(b)$. Continuing in this way and starting with c instead of b we will reach a contradiction to the d.c.c. Therefore $Z \subseteq Q$, by Lemma 2(i).

(ii) Assume $Q = Z$ and take any element $0 \neq a \in Q$. By assumption $a = bc$ for some b, $c \in Q$ (use Lemma 6). Hence $r(c) \subseteq r(a)$, and with the same arguments as in (i) we get $r(c) \subset r(a)$. This leads again to a contradiction as in (i). Therefore $Z \subset Q$. □

Now we are able to prove Theorem 8.

Proof. *Case* 1. Assume that R satisfies a.c.c. on principal right annihilators. By the symmetric version of Theorem 2.2 and the final remark in [6], $R/P(R)$ is a right nonsingular right D-ring which satisfies (C). Then $P(R)$ is completely prime by Proposition 3. Also, $Z = P(R)$ by Lemmas 1 and 4. Finally by [7, Theorem 3.2], we have that P is either nilpotent or $P = P^2 \neq 0$. Assume $P = P^2 \neq 0$ and take any $0 \neq a \in P$. Then there exists a_1, $b_1 \in P$ with $a = a_1 b_1$. Repeating the argument, starting with a_1 instead of a, we have $a = a_2 b_2 b_1$ for some a_2, $b_2 \in P$. By induction, we get a sequence $\{b_1, b_2, \dots\}$ of elements of P such that for every $m \geq 1$ there exists $a_m \in P$ with $a = a_m b_m \cdots b_1$. On the other hand $P = Z$ is right T-nilpotent, and so we get $a = 0$, a contradiction.

Case 2. Assume that R satisfies d.c.c. on principal right annihilators.

Since the generalized nil radical A is completely prime, $Z \subseteq A$ by Lemma 7. So $Z \subseteq P$ if $A = P$. If $A \neq P$ holds, we have $A = A^2 \neq 0$ by Lemma 2(iv) which implies $Z \subset A$ by Lemma 7 again. Thus, by Lemma 5, $Z \subseteq P$ follows in any case. Therefore by Lemma 4 $P = Z$ and R/P is a prime ring which has no nonzero nilpotent elements. Consequently P is completely prime. Finally, if P is not nilpotent, as in Case 1 we have $P = P^2 \neq 0$. So we get $Z \subset P$, a contradiction. □

Corollary 9. *Let R be a right D-ring which satisfies condition* (C) *and a.c.c. on principal right annihilators. Then $P = N_l(R)$, where $N_l(R)$ is the set of left zero-divisors of R.*

Proof. Obviously we have $P \subseteq N_l(R)$. Assume there exists $a \notin P$ with $r(a) \neq 0$. Hence $r(a) \subseteq r(a^2) \subseteq \cdots$ and $a^n \notin P$ for every integer n, since P is completely prime. By assumption, there exists m with $r(a^m) = r(a^{m+1})$. Take any $0 \neq b \in r(a)$. Thus $ab = 0$, and it follows that $b \in P \subset a^m R$; therefore, $b = a^m x$ for some $x \in R$, so $a^{m+1}x = 0$, which leads to $b = a^m x = 0$, a contradiction. □

We were unable to answer the following

Question. Is $P = N_l(R)$ also under d.c.c. for principal right annihilators?

Obviously, it has an affirmative answer if R is prime.

For the sake of completeness, we include a rather obvious example showing the relevance of assumption (C).

Example 10. There exist right D-rings in which the prime radical is not a completely prime ideal, even under strong conditions of finiteness.

Let K_i, $i = 1, 2, \ldots, n$, be fields and set $R = K_1 \oplus K_2 \oplus \cdots \oplus K_n$. We denote by e_i for $i = 1, \ldots, n$ the canonical idempotent $(0, \ldots, 1, \ldots, 0)$. It is easy to check that every ideal of R is of the type Re with $e = e_{i_1} + \cdots + e_{i_j}$, for some idempotents e_{i_k}. So the lattice of ideals of R is finite. By Theorem 1.6 in [11] R is right distributive if and only if for every $a, b \in R$ there exist $x, y \in R$ with $bx \in aR$, $ay \in bR$, and $x + y = 1$. Applying this result it can easily be deduced that the ring R constructed above is right (and left) distributive. We have $J(R) = (0)$ and so R does not satisfy the assumption (C), since (0) is neither completely prime nor prime. Thus the prime radical $P(R) = 0$ is not prime. Moreover, we remark that any nonzero ideal of R is idempotent and so there exist nonzero idempotent ideals, which are not prime provided $n \geq 3$. (We recall from [1] that in right chain rings idempotent ideals are always completely prime.)

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References

1. C. Bessenrodt, H. H. Brungs, and G. Törner, *Right chain rings, Part* 1, Schriftenreihe des Fachbereichs Mathematik, Universität Duisburg, 1990.
2. H. H. Brungs, *Rings with a distributive lattice of right ideals*, J. Algebra **40** (1976), 392–400.
3. N. I. Dubrovin, *An example of a nearly simple chain ring with nilpotent elements*, Mat. USSR-Sb. **120** (1983), 441–447. (Russian)
4. B. J. Gardner, *Some aspects of T-nilpotence*, Pacific J. Math. **53** (1974), 117–130.
5. K. R. Goodearl, *Ring theory, nonsingular rings and modules*, Marcel Dekker, New York, 1976.
6. B. Johns, *Chain conditions and nil ideals*, J. Algebra **73** (1981), 287–294.
7. R. Mazurek, *Distributive rings with Goldie dimension one*, Comm. Algebra **19** (1991), 931–944.
8. R. Mazurek and E. R. Puczyłowski, *On nilpotent elements of distributive rings*, Comm. Algebra **18** (1990), 463–471.
9. E. Posner, *Left valuation rings and simple radical rings*, Trans. Amer. Math. Soc. **107** (1963), 458–465.
10. M. Schröder, *Über N. I. Dubrovin's Ansatz zur Konstruktion von nicht vollprimen Primidealen in Kettenringen*, Results in Math. **17** (1990), 296–306.
11. W. Stephenson, *Modules whose lattice of submodules is distributive*, Proc. London Math. Soc. (3) **28** (1974), 291–310.
12. A. A. Tuganbaev, *Rings in which the structure of right ideals is right distributive*, Soviet Math. (Iz. VUZ) **30** (1986), 58–65.

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