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ON THE SPECTRUM OF THE NEUMANN LAPLACIAN OF LONG RANGE HORNS: A NOTE ON THE DAVIES-SIMON THEOREM

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ABSTRACT. For a class of regions with cusps (e.g., $\Omega = \{(x, y) : x > 1, |y| < \exp(-x^\alpha)\}$, $0 < \alpha < 1$) we show that $\sigma_{\mathrm{ac}}(-\Delta_N^\Omega) = [0, \infty)$ of uniform multiplicity one, $\sigma_{\mathrm{sing}}(-\Delta_N^\Omega) = \varnothing$, and $\sigma_{\mathrm{pp}}(-\Delta_N^\Omega)$ consists of a discrete set of embedded eigenvalues of finite multiplicity.

1. INTRODUCTION

This note is a contribution to the study of spectral properties of Neumann Laplacians on regions of the form

$$\Omega = \{(x, y) : x > 1,\ |y| < f(x)\}, \tag{1.1}$$

where f is a strictly positive $C^2[1, \infty)$ function. The Neumann Laplacian on Ω, $-\Delta_N^\Omega = H$, is a unique selfadjoint operator in $L^2(\Omega)$ whose quadratic form is given by the closure of

$$q(\phi, \phi) = \int_\Omega |\nabla\phi|^2\, dx, \tag{1.2}$$

on $C_0^2(\overline{\Omega})$. The spectral properties of H in regions (1.1) have been studied in [2, 5, 8] and, if specialized to the case when $f(x) = \exp(-x^\alpha)$, $\alpha > 0$, they can be stated as follows ($f \sim g$ stands for $\lim_{x\to\infty} f(x)/g(x) = 1$).

Theorem 1.1. (i) [5] *H has a discrete spectrum iff $\alpha > 1$.*

(ii) [8] *If $N_E(H)$ denotes the number of eigenvalues of H which are less than E, then for $\alpha > 1$*

$$N_E(H) \sim E^{1/2+1/(2(\alpha-1))}C(\alpha) + E\int_1^\infty \exp(-x^\alpha)\, dx, \tag{1.3}$$

where

$$C(\alpha) = \frac{1}{4(\alpha-1)\sqrt{\pi}}\left(\frac{\alpha}{2}\right)^{1/(1-\alpha)}\frac{\Gamma(1/(2(\alpha-1)))}{\Gamma(3/2+1/(2(\alpha-1)))}.$$

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(iii) [2] *If* $\alpha = 1$, $\sigma_{\text{ac}}(H) = [\frac{1}{4}, \infty)$ *of uniform multiplicity one,* $\sigma_{\text{sing}}(H) = \varnothing$, *and* $\sigma_{\text{pp}}(H)$ *consists of a discrete set* $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_n \to \infty$ *of eigenvalues of finite multiplicity.*

(iv) [2] *If* $0 < \alpha < \frac{1}{2}$, *the results of part* (iii) *remain valid except that now* $\sigma_{\text{ac}}(H) = [0, \infty)$.

In this paper we prove a theorem which, specialized to the above example, covers the range $\frac{1}{2} \leq \alpha < 1$; namely, we will show that all the conclusions of Theorem 1.1(iv) are valid for $0 < \alpha < 1$. Thus, at least in the above special case, we have a complete picture of what happens with the Neumann Laplacian (for some recent examples of surprising spectral properties of $-\Delta_N^{\Omega}$, even for *bounded regions* Ω, we refer to [6, 14]).

Our strategy (and our proof) follows closely the one of Davies and Simon [2]. The major observation made in [2] is that any spectral behavior which distinguishes H from the Dirichlet case (e.g., existence of a continuous spectrum) can come only from the subspace P consisting of functions which depend on the x variable only and which are in the form domain of H (such functions *cannot* be in the form domain of Dirichlet Laplacian). Restricted to such a subspace, the form (1.1) acts as

$$2\int_1^{\infty} |u'(x)|^2 f(x)\, dx,$$

and viewed as a form on $L^2([1, \infty), 2f(x)\, dx)$ yields an operator which is unitarily equivalent to the one-dimensional Schrodinger operator on $L^2([1, \infty), dx)$ of the form

$$H_V = -\frac{d^2}{dx^2} + V(x), \qquad V = \frac{1}{4}\left(\frac{f'}{f}\right) + \frac{1}{2}\left(\frac{f'}{f}\right)'. \tag{1.4}$$

On the subspace orthogonal to P, H acts in effect as a Dirichlet Laplacian, and off-diagonal perturbation terms are controlled by the function

$$k(x) = |f'(x)| + f'(x)^2/f(x).$$

We say that a function g on $[1, \infty)$ is short-range if $g(x) = O(x^{-1-\varepsilon})$ for some $\varepsilon > 0$. Davies and Simon proved the following in [2]:

Theorem 1.2. (i) *If* $f \to 0$ *and* $k \to 0$ *as* $x \to \infty$, H *has discrete spectrum if and only if* H_V *does.*

(ii) *If* k *and* V *are short range, we have* $\sigma_{\text{ac}}(H) = [0, \infty)$ *of uniform multiplicity,* $\sigma_{\text{sing}}(H) = \varnothing$, *and* $\sigma_{\text{pp}}(H)$ *consists of a discrete set* $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_n \to \infty$ *of embedded eigenvalues of finite multiplicity.*

When $f(x) = \exp(-x^{\alpha})$,

$$V(x) = \frac{\alpha^2}{4}x^{2(\alpha-1)} - \frac{\alpha(\alpha-1)}{2}x^{(\alpha-2)}$$

and Theorem 1.2 certainly yields parts (i), (iv) of Theorem 1.1. Case $\alpha = 1$ is somewhat special since then $V = \frac{1}{4}$: The argument of [2] still carries through, with the obvious shift of the essential spectrum. With some additional technical assumptions it was shown in [8] that in case (i) of Theorem 1.1 one has

$$N_E(H) \sim N(H_V) + \frac{E}{2}\operatorname{Vol}(\Omega). \tag{1.5}$$

Using the semiclassical formula and (1.5) we get part (ii) of Theorem 1.1.

It is now easy to understand why $\alpha = \frac{1}{2}$ is the critical value: while for $0 < \alpha < \frac{1}{2}$ the potential V is short-range, for $\frac{1}{2} \leq \alpha < 1$ it becomes long-range (thus the name, long-range horns), and to treat perturbations one has to use the long-range scattering theory, which tends to be technically involved (particularly in the case when $V(x) \sim x^{-\beta}$, $0 < \beta < \frac{1}{2}$, which corresponds to the range $\frac{3}{4} \leq \alpha < 1$). The authors of [2] used the short-range Enss theory to treat perturbations and conjectured "that it is likely that one can modify ... [their] ... argument" to prove the analog of Theorem 1.2(ii) in the long-range case. By replacing the short-range Enss theory with the long-range one we extend Theorem 1.2 to the case where V is long-range and dilation analytic. The dilation analyticity assumption, although restrictive, covers the example of Theorem 1.1 and enables a particularly simple technical treatment, thanks to the work of Perry [10, 11].

In the sequel, H_0 stands for the one-dimensional free Laplacian. Let $u(\theta)$: $L^2(\mathbf{R}) \to L^2(\mathbf{R})$ be a unitary mapping defined as

$$u(\theta)\phi(x) = \exp(\theta/2)\phi(\exp(\theta)x).$$

A potential $\widehat{V}$, defined on the whole real line, is called dilation analytic [11, 12] if $\widehat{V}$ is H_0-compact and the operator-valued function

$$C(\theta) = u(\theta)\widehat{V}u(\theta)^{-1}(H_0 + 1)^{-1}$$

extends to an analytic operator-valued function on the strip $\mathscr{S}(a) = \{z : -a < \operatorname{Im}(z) < a\}$ for some $a > 0$.

With V as in (1.4), we define

$$\widehat{V}(x) = V(|x| + 1), \qquad x \in \mathbf{R}.$$

In the sequel we assume that f is $C^3[1, \infty)$ so that V is a differentiable function. Our main result is

Theorem 1.3. *Suppose that k and V' are short range and that $\widehat{V}$ is dilation analytic. Then all conclusions of Theorem* 1.1(iv) *remain valid.*

Since, for $f(x) = \exp(-x^\alpha)$,

$$\widehat{V}(x) = \frac{\alpha^2}{4}(|x| + 1)^{2(\alpha-1)} - \frac{\alpha(\alpha - 1)}{2}(|x| + 1)^{\alpha-2}$$

is certainly dilation analytic [11] if $0 < \alpha < 1$, the above theorem applies to the range of α not covered by Theorem 1.1.

We finally remark that all the above-mentioned results have natural extensions to higher-dimensional nonsymmetric regions, as well as to manifolds with metric cusps at infinity [2, 6, 8].

2. Proof of Theorem 1.3

Let H_V be the operator (1.4) with the Neumann boundary condition at 1 and denote by

$$D = \{\psi : \psi \in C^2[1, \infty),\ \psi'(1) = 0\}$$

its domain of essential selfadjointness. Let $J\colon L^2(R) \to L^2(\Omega)$ be the embedding $(J\phi)(x) = \phi(x)/\sqrt{2f(x)}$, and denote $Q = 1 - JJ^*$.

The first ingredient in our argument is the following result of Davies and Simon [2].

Theorem 2.1. (i) *If* $\phi \in D$,

$$\|(H+1)^{-1/2}(HJ - JH_V)\phi\| \leq 2 \cdot \|f'\phi\|_2 + \|((f')^2/f)\phi\|_2 + C|\phi(1)|,$$

where C *is a positive constant.*

(ii) $Q(H+1)^{-1/2}$ *is a compact operator in* $L^2(\Omega)$.

(iii) *If* g *is a continuous function on* $R \cup \{\infty\}$, $g(H)J - Jg(H_V)$ *is a compact operator in* $L^2(\Omega)$.

We now use the decay of V' and the dilation analyticity of $\widehat{V}$ to obtain a useful decomposition of V into a short-range part and a smooth long-range dilation analytic part [10]. $\widehat{V}$ can be decomposed as

$$\widehat{V}(x) = \widehat{V}_L(x) + \widehat{V}_S(x), \tag{2.1}$$

where $\widehat{V}_L(x)$ is a C^∞ dilation analytic potential with all derivatives bounded,

$$\widehat{V}'(x) = O(|x|^{-1-\varepsilon}),$$

and $\widehat{V}_S(x)$ is a short-range function. Both parts $\widehat{V}_L$, $\widehat{V}_S$ can be chosen to be even functions around 0, and thus they induce the decomposition

$$V(x) = V_L(x) + V_S(x),$$

with obvious inheritance of properties. We refer to [10] for a proof of (2.1). $\widehat{V}_L$ is defined as the Weierstrass transform of $\widehat{V}$,

$$\widehat{V}_L(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4}\right) \widehat{V}(y)\, dy.$$

Let

$$\widehat{H}_L = H_0 + \widehat{V}_L, \qquad H_L = -\frac{d^2}{dx^2} + V_L,$$

where the second operator acts on $L^2([1, \infty), dx)$ with the Neumann boundary condition at 1. The third ingredient in the argument consists of Perry's propagation estimates [10, 11] on $\exp(-it\widehat{H}_L)$ (see also [1]), which automatically yield the propagation estimates on $\exp(-itH_L)$. Let A be a scale transformation around $x = 1$ [1, 10],

$$A = \tfrac{1}{2}((x-1)\cdot p + p \cdot (x-1)),$$

where $p = -iD = -i\partial/\partial x$ in the x-space representation. A is essentially selfadjoint on $C_0^\infty(R)$, $\sigma(A) = \sigma_{\mathrm{ac}}(A) = (-\infty, \infty)$, and A leaves invariant the subspace of functions which are even around 1. By $P_\pm$ we denote the spectral projections of A on $\pm(0, \infty)$ and by $P_\pm^a$ the spectral projections on $\pm(a, \infty)$. Obviously, $P_+ + P_- = 1$. It is a standard result [12] that operators $\widehat{H}_L$, H_L have no (strictly) positive eigenvalues, that $\sigma_{\mathrm{sing}}(\widehat{H}_L) = \sigma_{\mathrm{sing}}(H_L) = \varnothing$, and that $\sigma_{\mathrm{ac}}(\widehat{H}_L) = \sigma_{\mathrm{ac}}(H_L) = [0, \infty)$. Since $\widehat{V}$ is even around 0, $\widehat{H}_L$ preserves the subspace of functions which are in its domain and are even around 0. After translation by 1, the restriction of $\widehat{H}_L$ to that subspace coincides with H_L. In the sequel we will use Enss' notation where $F(M)$ stands for the characteristic function of the set M. P_{ac} will denote the spectral projection on the absolutely continuous subspace of H_L. Viewing H_L as a restriction of $\widehat{H}_L$ (keeping the above translation in mind) we have

Theorem 2.2. (i) *Let g be a C^∞ function with support in $[\alpha, \beta]$, $\alpha > 0$. Then for any $\delta > 0$, $N > 0$, and $a \in \mathbf{R}$,*

$$\|F(1 \le x < |t|^{1-\delta}) \exp(itH_L) g(H_L) P_\pm^a\| = O(|t|^{-N}) \tag{2.2}$$

as $t \to \pm\infty$.

(ii) $\operatorname{s-lim}_{t\to\pm\infty} P_\pm^a \exp(-itH_L) = 0$.

The above theorem is a result by Perry [10, 11].

Since H_L plays the role of the "free Laplacian" in the discussion below, it is worthwhile to rephrase part (i) of Theorem 2.1 as

$$\|(H+1)^{-1/2}(JH - JH_L)\phi\| \le 2 \cdot \|f'\phi\|_2 + \|((f')^2/f)\phi\|_2 + \|V_L\phi\|_2 + C|\phi(1)|$$

and to note, concerning part (iii), that $g(H)J - Jg(H_L)$ is a compact operator in $L^2(\Omega)$ (since $g(H_V) - g(H_L)$ is compact).

Since V_L is bounded from below, without loss of generality we can assume $V_L \ge 0$. The final technical lemma is the following

Lemma 2.3.

$$\|(H+1)^{-1/2}(HJ - JH_L)(H_L+1)^{-1}F(x > R)\| = O(R^{-1-\varepsilon}).$$

The proof is standard [1, Lemma 5.4], using Theorem 2.1(i) and the fact that k, V_S are short-range.

From now on, the proof follows line-by-line the argument of Davies and Simon [2]. We give the details in the appendix for the reader's convenience.

Appendix

Step 1. $\operatorname{s-lim}_{t\to\pm\infty}(H+1)^{-1/2}\exp(itH)J\exp(-itH_L)P_{\text{ac}}$ exists. Let us consider only the limit $t \to \infty$; a similar consideration applies to the other one. By Cook's criterion, it suffices to show that

$$\int_0^\infty \|(H+1)^{-1/2}(HJ - JH_L)\exp(-itH_L)g(H_L)P_+^a\|\, dt < \infty,$$

where g is as in Theorem 2.2, since $\bigcup_{a,g} \operatorname{Ran} g(H_L)P_+^a$ is dense in $\mathscr{H}_{\text{ac}}(H_L)$. Denote

$$A(t) = \|(H+1)^{-1/2}(HJ - JH_L)\exp(-itH_L)g(H_L)P_+^a\|.$$

Choosing $\delta < \varepsilon$ in (2.2), we estimate, using Theorem 2.2 and Lemma 2.3,

$$\begin{aligned} A(t) &\le \|(H+1)^{-1/2}(HJ - JH_L)(H_L+1)^{-1}F(x \ge t^{-1-\delta})\| \\ &\quad + \|F(1 \le x < t^{-1-\delta})\exp(-itH_L)(H_L+1)g(H_L)P_+^a\| \\ &\le O(t^{-1-\varepsilon+\delta}). \end{aligned}$$

Step 2. The wave operators $\Omega^\pm = \operatorname{s-lim}_{t\to\mp\infty}\exp(itH)J\exp(-itH_L)P_{\text{ac}}$ exist. $(H+1)^{-1/2}J - J(H_L+1)^{-1/2}$ is compact by Theorem 2.1, since

$$\begin{aligned} &(H+1)^{-1/2}J - J(H_L+1)^{-1/2} \\ &\quad = (H+1)^{-1/2}J - J(H_V+1)^{-1/2} + J(H_V+1)^{-1/2} - J(H_L+1)^{-1/2}. \end{aligned}$$

Thus

$$\text{s-}\lim_{t\to\mp\infty} \exp(-itH)((H+1)^{-1/2}J - J(H_L+1)^{-1/2})\exp(-itH_L)P_{\mathrm{ac}} = 0\,.$$

By Step 1, $\Omega^{\pm}\phi$ exist if $\phi \in \operatorname{Ran}(H_L+1)^{-1/2}P_{\mathrm{ac}}$, and the last set coincides with $\mathscr{H}_{\mathrm{ac}}(H_L)$.

Step 3. $(H+1)^{-1/2}(\Omega^{\pm} - J)g(H_L)P_{\pm}$ is a compact operator.

First, since $(H_L+1)^{-1/2}(\exp(itH_L)J\exp(-itH_L) - J)g(H_L)P_{\pm}$ converges in the operator norm as $t \to \pm\infty$, it suffices to show that each of these operators is compact. On the other hand, they are integrals of operators of the form

$$(H+1)^{-1/2}(HJ - JH_L)\exp(-itH_L)g(H_L)P_{\pm}\,, \tag{A.1}$$

and thus it suffices to show that the operators (A.1) are compact. But that is a consequence of part (iii) of Theorem 2.1 and of the fact that V_S is short-range.

Step 4. If $\phi_n \in \mathscr{H}_{\mathrm{ac}}(H)^{\perp}$, with $\|(H+1)\phi_n\|$ bounded and $(H+1)^{1/2}\phi_n \to 0$ weakly, then $\|\phi_n\| \to 0$ as $n \to \infty$.

First, $(\Omega^{\pm})^*\phi_n = 0$ for all n, and thus for any $C_0^{\infty}((0,\infty))$ function g, Step 3 yields

$$P_{\pm}g(H_L)J^*\phi_n \to 0\,.$$

Since $P_+ + P_- = 1$ and $g(H_L)J^* - J^*g(H)$ is a compact operator, we obtain that $J^*g(H)\phi_n \to 0$. Since $\|(H+1)\phi_n\|$ is bounded, we estimate

$$\|g(H)\phi_n - \phi_n\| \le C \cdot \|(g(H)-1)\cdot(H+1)^{-1}\|\,. \tag{A.2}$$

Since g is arbitrary, the left-hand side of (A.2) can be made arbitrarily small, and thus $J^*\phi_n \to 0$, and so $JJ^*\phi_n \to 0$. By Theorem 2.1, $Q\phi_n = (1-JJ^*)\phi_n \to 0$, and we conclude that $\phi_n \to 0$.

Step 5. $\sigma_{\mathrm{sing}}(H) = \varnothing$ and in any finite interval H has only finitely many eigenvalues. If any of the statements is not valid, we can construct an orthonormal sequence ϕ_n so that $\phi_n \in \mathscr{H}_{\mathrm{ac}}(H)^{\perp}$, $\|(H+1)\phi_n\|$ is bounded, and $(H+1)^{1/2}\phi_n \to 0$ weakly. Step 4 implies then that $\phi_n \to 0$, which contradicts the fact that the sequence ϕ_n is orthonormal.

Step 6. $\sigma_{\mathrm{ac}}(H)$ has multiplicity one. It suffices to show that $\operatorname{Ran}\Omega^+ = \mathscr{H}_{\mathrm{ac}}(H)$. Suppose that it is not, namely, that we can find a nonzero vector $\phi \in \mathscr{H}_{\mathrm{ac}}(H) \cap (\operatorname{Ran}\Omega^+)^{\perp} \cap D(H)$. Define $\phi_n = \exp(-inH)\phi$. Part (ii) of Theorem 2.2 yields

$$P_{\pm}(\Omega^{\pm})^*\phi_n = P_{\pm}\exp(-inH_L)P_{\mathrm{ac}}(\Omega^{\pm})^*\phi \to 0 \quad \text{as } n \to \infty\,.$$

Thus, as in Step 4, $\|\phi_n\| \to 0$, and we derive that $\phi = 0$, a contradiction.

Step 7. There exists a discrete set of embedded eigenvalues. This is a consequence of symmetry of Ω with respect to the axes $y = 0$. Let E be the subspace of $L^2(\Omega)$ consisting of those functions which are even under reflection $(x, y) \to (x, -y)$. That subspace is invariant under Q and H, and thus H restricted to it has a compact resolvent by Theorem 2.1.

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