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A REPRESENTATION LATTICE ISOMORPHISM FOR THE PERIPHERICAL SPECTRUM

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ABSTRACT. In this paper we construct a representation isometric lattice isomorphism for the peripherical spectrum of a positive operator on a Banach lattice. By a representation lattice homomorphism, we mean that the peripherical spectrum of the operator is identified with the spectrum of the induced isometric lattice homomorphism. A simple proof of a "zero-two" law follows easily from our representation technique.

1. The representation lattice homomorphism

We develop our technique in the context of Banach lattices with p-additive norm. Following Zaanen [Zaa],

1.1. **Definition.** Let $1 \leq p < \infty$. A Banach lattice E for which $\|x+y\|^p = \|x\|^p + \|y\|^p$ whenever $x \wedge y = 0$ is called an *abstract $\mathscr{L}^p$-space* (or an *AL_p-space*).

In fact, the norm of an AL_p-space $(1 \leq p < \infty)$ is p-superadditive for all positive elements. In other words, if E is an AL_p-space, then

$$\|x+y\|^p \geq \|x\|^p + \|y\|^p \quad \forall x, y \geq 0.$$

As a straightforward consequence of the p-superadditivity of the norm, we get the following basic property of AL_p-spaces:

(1) If T is a contraction on E that verifies $0 \leq x \leq Tx$ for some $x \in E$, then $Tx = x$.

Now let E be an AL_p-space $(1 \leq p < \infty)$, and let $\mathscr{F}$ denote a free ultrafilter on $\mathbf{N}$. The $\mathscr{F}$-product $\widehat{E}_{\mathscr{F}}$ is actually an AL_p-space (see [S1, Chapter V, §1]). We denote by P the following isometric lattice isomorphism on $\widehat{E}_{\mathscr{F}}$:

$$P((x_1, x_2, \dots) + c_{\mathscr{F}}(E)) = (x_2, x_3, \dots) + c_{\mathscr{F}}(E),$$

whose inverse isometric lattice homomorphism is given by

$$Q((x_1, x_2, \dots) + c_{\mathscr{F}}(E)) = (0, x_1, x_2, \dots) + c_{\mathscr{F}}(E).$$

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1.2. **Definition.** Let T be a positive contraction on E, and denote by $\widehat{T}_{\mathscr{F}}$ its canonical extension to the $\mathscr{F}$-product $\widehat{E}_{\mathscr{F}}$. We define the *limit space of* T as the Banach space $E(T) = \operatorname{Ker}(P - \widehat{T}_{\mathscr{F}})$ and the *limit operator of* T by $\widetilde{T} = \widehat{T}_{\mathscr{F}}|_{E(T)} = P|_{E(T)}$. The fact that

$$\{T^n x\} + c_{\mathscr{F}}(E) \in E(T) \quad \forall x \in E$$

justifies our terminology. The Banach subspace

$$A(T) = \overline{\{\{T^n x\} + c_{\mathscr{F}}(E) : x \in E\}} \subseteq E(T)$$

will be called the *asymptotic space* of T.

As an immediate consequence of property (1), $E(T) = \operatorname{Ker}(I - Q\widehat{T}_{\mathscr{F}})$ is in fact a sublattice of $\widehat{E}_{\mathscr{F}}$ and so an AL_p-space. Moreover, as $\widetilde{T}$ is the restriction of an isometric lattice homomorphism, $\widetilde{T}$ is obviously an isometric lattice homomorphism.

We can now state our basic lemma, which translates the techniques of Allan-Ransford [A] and Phong-Lyubich [L] to the setting of Banach lattices.

1.3. **Theorem.** *Let T be a positive contraction on the AL_p-space E with $1 \leq p < \infty$, and let $\widetilde{T}$ denote its limit operator. Then we have*

$$\Gamma \cap \sigma_p(T) \subseteq \sigma(\widetilde{T}) \subseteq \Gamma \cap \sigma(T), \quad \textit{where } \Gamma = \{\lambda \in \mathbf{C} : |\lambda| = 1\}.$$

Proof. Let $\lambda \in \rho(T)$. As

$$\widehat{R(\lambda, T)}(E(T)) \subseteq E(T),$$

we get $\lambda \in \rho(\widetilde{T})$, where the resolvent is given by

$$R(\lambda, \widetilde{T}) = \widehat{R(\lambda, T)}|_{E(T)}.$$

But $\widetilde{T}$ is an invertible isometry and so the inclusion $\sigma(\widetilde{T}) \subseteq \Gamma \cap \sigma(T)$ is already proved.

On the other hand, if $\lambda \in \Gamma \cap \sigma_p(T)$, there exists a nonzero vector x with $Tx = \lambda x$. Defining now

$$\tilde{x} = (x, \lambda x, \lambda^2 x, \ldots) + c_{\mathscr{F}}(E) \in E(T),$$

we obtain $\widetilde{T}\tilde{x} = \lambda \tilde{x}$, $\|\tilde{x}\| = \|x\| \neq 0$; therefore, $\lambda \in \sigma(\widetilde{T})$.

Applications

Let T be a positive contraction on L^1. In 1970 Orstein and Sucheston [O] showed that

$$\text{(1)} \qquad \sup_{\|f\|_1 \leq 1} \lim_{n \to \infty} \|T^n f - T^{n+1} f\|_1$$

is either 0 or 2. This surprising result opened a new direction of research. Wittmann [W] extended this "zero-two" law to AL_p-spaces. On the other hand, Zaharopol [Zah], Katznelson-Tzafriri [K], and Schaefer [S2] proved that, given a positive linear contraction in an arbitrary Banach lattice, the limit $\lim_n |||T^n - T^{n+1}|||$ is either 0 or 2.

We now deduce a simple proof of a uniform "zero-two" law from the above representation technique. We need the following modification of an Arendt-Schaefer-Wolff result (see [Ar, Lemma 3.3]):

2.1. **Lemma.** *Let T be a positive isometry on the Banach lattice E, and suppose that $r(I - T) < \sqrt{3}$. Then we have $T = I$.*

Proof. By the classical result of Gelfand (see [A]), we only need to show $\sigma(T) = 1$. If this is not verified, as $\sigma(T)$ is cyclic (see [S1]), there must be an element $a \in \sigma(T)$ such that $\frac{2}{3}\pi \le \arg a \le \frac{4}{3}\pi$ holds; this implies $-1 \le \mathrm{Re}(a) \le -\frac{1}{2}$. From this inequality we obtain

$$r(I - T)^2 \ge |a - 1|^2 = |a|^2 - 2\mathrm{Re}(a) + 1 \ge 3.$$

2.2. **Theorem.** *Let E be an AL_p-space $(1 \le p < \infty)$, and let T be a positive contraction on E. Then the following statements are equivalent:*

(a) $\lim_{n\to\infty} \|T^n - T^{n+1}\| = 0$.
(b) $\lim_{n\to\infty} \|T^n - T^{n+1}\| < \sqrt{3}$.

Proof. (b) $\to$ (a) Given a free ultrafilter $\mathcal{F}$ on E, if we denote by $S = \widehat{T}_{\mathcal{F}}$ the canonical extension of T to the $\mathcal{F}$-product AL_p-space $\widehat{E}_{\mathcal{F}}$, we have (see [S1, Chapter V, §1])

$$\sigma_{ap}(T) = \sigma_p(S), \qquad \|T^n - T^{n+1}\| = \|S^n - S^{n+1}\|.$$

By the Katznelson-Tzafriri theorem [K], it suffices to prove $\sigma_p(S) \cap \Gamma = \sigma(T) \cap \Gamma \subseteq \{1\}$. However, given $x \in E(T)$, denoting by $\widetilde{S}$ the limit operator of S we get

$$\|x - \widetilde{S}x\| = \|Q^n(\widehat{S}^n - \widehat{S}^{n+1})x\| \le \|S^n - S^{n+1}\| \, \|x\|$$

and so we deduce $\|I - \widetilde{S}\| < \sqrt{3}$. Lemma 2.1 now shows that $\widetilde{S} = I$, and then from Theorem 1.3 we conclude

$$\Gamma \cap \sigma_p(S) \subseteq \sigma(\widetilde{S}) = \{1\}.$$

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