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ARENS REGULARITY AND THE $A_p(G)$ ALGEBRAS

BRIAN FORREST

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ABSTRACT. Let G be a locally compact group. Let $A_p(G)$ denote the Herz algebra associated with $1 < p < \infty$. We show that, for a large class of groups which contains every commutative group, $A_p(G)$ is Arens regular if and only if G is finite.

1. INTRODUCTION

Let G be a locally compact group. If $1 < p < \infty$, we let $A_p(G)$ denote the subspace of $C_0(G)$ consisting of functions of the form $u(x) = \sum_{i=1}^{\infty}(f_i * \tilde{g}_i)(x)$ where $f_i \in L_p(G)$, $g_i \in L_q(G)$, $1/p + 1/q = 1$, $\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty$, $f(x) = f(x^{-1})$, and $\tilde{f}(x) = \overline{f(x^{-1})}$. $A_p(G)$ is a commutative Banach algebra with respect to pointwise operations and the norm

$$\|u\|_{A_p(G)} = \inf\left\{\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q | u = \sum_{i=1}^{\infty}(f_i * \tilde{g}_i)\right\}.$$

In case $p = 2$, $A_2(G) = A(G)$ is the Fourier algebra of G, which was introduced for noncommutative groups by Eymard [2]. For $p \neq 2$, $A_p(G)$ was first studied by Herz in [5].

If $1 < p < \infty$, then $PM_p(G)$ denotes the closure of $L_1(G)$, considered as an algebra of convolution algebras on $L_p(G)$, with respect to the weak operator topology in $B(L_p(G))$. $PM_p(G)$ can be identified with the dual of $A_p(G)^*$.

We will show that if G is a discrete commutative group, then $PM_p(G)$ is separable if and only if G is finite. As a consequence of this result, we are able to show that if $A_p(G)$ is a regular Banach algebra, then every abelian subgroup of G is finite. This extends a number of results of [3] and answers a question raised in [7] for a large class of groups including all locally compact abelian groups.

2. ARENS REGULARITY AND $A_p(G)$

When $p = 2$, $A_2(G) = A(G)$ is the unique predual of the von Neumann algebra $PM_2(G) = VN(G)$. It is well known that a von Neumann algebra is

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separable if and only if it is finite dimensional. It follows that $PM_2(G)$ is separable if and only if G is finite. While it is reasonable to believe that this is true for all p with $1 < p < \infty$, we do not know how to prove this. We do, however, have the following:

Theorem 1. *Let G be a discrete commutative group. If $PM_p(G)$ is separable, then G is finite.*

Proof. First we assume that G contains an element x of infinite order. Let $H = \langle x \rangle$. Then H is isomorphic with $\mathbb{Z}$, so $\widehat{H} \cong \Pi$.

Decompose $[-\pi, \pi)$ dyadically into the family

$$I_j = \begin{cases} [-\pi, -\pi/2) \cup [\pi/2, \pi) & \text{if } j = 0, \\ [\pi/2^{i+1}, \pi/2^i) & \text{if } j > 0, \\ (-\pi/2^{|j|}, -\pi/2^{|j|+1}] & \text{if } j < 0, \end{cases}$$

where $j \in \mathbb{Z}$.

Let $\Delta_j = \{e^{it} | t \in I_j\}$. Then the dyadic decomposition $\{\Delta_j\}_{j \in \mathbb{Z}}$ has the Littlewood-Paley Property (see [1, p. 135]). Therefore, if φ is a bounded function on Π, which is constant on each of the Δ_j's, then $\widehat{\varphi} \in PM_p(H)$. Moreover, $\|\varphi\|_\infty \leq \|\widehat{\varphi}\|_{PM_p}(H) \leq M\|\varphi\|_\infty$ for some M which is independent of φ. If we let $\Gamma = \{\varphi | \varphi = \sum_{j \in \mathbb{Z}} a_j \chi_{\Delta_j}, \ a_j = 0, 1\}$ (where χ_{Δ_j} denotes the characteristic function of Δ_j), then Γ is uncountable and $\|\widehat{\varphi}_1 - \widehat{\varphi}_2\|_{PM_p(H)} \geq 1$ whenever $\varphi_1, \varphi_2 \in \Gamma$, $\varphi_1 \neq \varphi_2$. It follows that $PM_p(H)$ is not separable, but since H is open, $PM_p(H)$ is a closed subspace of $PM_p(G)$. Consequently, G must be torsion.

Let H be a countably infinite subgroup of the torsion group G. Let $H_0 = \langle e \rangle$. Choose $x_1 \neq e$, $x_1 \in H$, and let $H_1 = \langle x_1 \rangle$. Since H is torsion, there exists $x_2 \notin H_1$, $x_2 \in H$. We let $H_2 = \langle x_1, x_2 \rangle$. We continue in this manner to construct a sequence $\{H_n\}_{n=0}^\infty$ of finite subgroups of H such that $H_n \subsetneq H_{n+1}$. Moreover, since H is countable, we can assume that $H = \bigcup_{n=0}^\infty H_n$. Let $K = \widehat{H}$. Let $K_n = H_n^\perp$. Then K is compact and, since $(K/K_n)\widehat{\ } = H_n$ is finite, each K_n is open in K. Moreover, $K_n \supsetneq K_{n+1}$ and $K_0 = K$. Let $\{\Delta_n\}_{n=0}^\infty = \{K_n \setminus K_{n+1}\}_{n=0}^\infty$. Then $\{\Delta_n\}_{n=0}^\infty$ is again a decomposition of K with the Littlewood-Paley Property [1, p. 68]. Just as before, it follows that $PM_p(H)$ is nonseparable, which is again impossible. Since G has no countably infinite subgroups, G must be finite. □

When G is an amenable group, it is known that $A(G)$ is Arens regular if and only if G is finite [8]. We are now able to establish a related result for $1 < p < \infty$.

Theorem 2. *Let G be a locally compact group. If $A_p(G)$ is Arens regular, then every abelian subgroup of G is finite. In particular, if G is abelian, then $A_p(G)$ is Arens regular if and only if G is finite.*

Proof. If $A_p(G)$ is Arens regular, then G is discrete [3, Theorem 3.2]. Let H be a countably infinite commutative subgroup of G. Then $A_p(H)$ is also Arens regular [3, Lemma 3.1]. It follows from [3, Proposition 3.11] that $PM_p(H)$ is separable. Hence, by Theorem 1, H is finite.

If G is finite, the reflexive algebra $A_p(G)$ is clearly Arens regular. □

We have shown that if $A_p(G)$ is Arens regular, then G is a discrete torsion group. Perhaps the most significant class of torsion groups is the class of locally finite groups. We show that for this class and indeed for a number of large classes of groups, $A_p(G)$ is Arens regular if and only if G is finite (see [3, Proposition 3.9]).

Proposition 3. *Let G be a discrete group which satisfies any of the following conditions:*

(i) *G is locally finite;*
(ii) *G is an elementary group;*
(iii) *G is locally nilpotent;*
(iv) *G is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{F})$ for some n and any field $\mathbb{F}$;*
(v) *G is a 2-group;*
(vi) *G is hyperfinite;*
(vii) *G is hypercentral;*
(viii) *G has an involution x with $|C_G(x)| \leq \infty$.*

Then $A_p(G)$ is Arens regular if and only if G is finite.

Proof. (i) If G is an infinite locally finite group, then G has an infinite abelian subgroup [6, Corollary 2.5]. If follows from Theorem 2 that G must be finite.

(ii) Since G is torsion, if G is elementary, then G is locally finite [9, §3.11].

(iii) If G is locally nilpotent, then either G is locally finite or G contains an infinite nilpotent subgroup. In any case, if G is infinite, then G has an infinite commutative subgroup.

The remaining cases can be established just as in the proof of [3, Proposition 3.9]. □

Let $\Phi = \{\delta_x | x \in G\} \subset PM_p(G)$, where $\delta_x(u) = u(x)$. Recently, Lau and Ülger showed that if G is amenable then $A_p(G)$ is Arens regular if and only if $PM_p(G) = \overline{\operatorname{span}}\,\Phi$ where the closure is with respect to the norm topology [7, Corollary 8.3]. They state, however, that they were unable to show that $PM_p(G) = \overline{\operatorname{span}}\,\Phi$ implies that G is finite. We have

Proposition 4. *Let G be a locally compact group for which $PM_p(G) = \overline{\operatorname{span}}\,\Phi$. Then G is discrete and G contains no infinite commutative subgroup.*

Proof. Observe that if $\overline{\operatorname{span}}\,\Phi = PM_p(G)$, then $A_p(G)$ is Arens regular even if G is nonabelian since each δ_x is clearly weakly almost periodic. The result now follows immediately from Theorem 2. □

It is clear that if G is in any of the classes included in the statement of Proposition 3, then $PM_p(G) = \overline{\operatorname{span}}\,\Phi$ if and only if G is finite. We are, however, still unable to establish this result for all G.

In the statement of Theorem 1, G is assumed to be discrete. We show that this was, in fact, not necessary. For any G, if $PM_p(G)$ is separable, then since it is also a dual space, $PM_p(G)$ has the Radon-Nikodým Property (RNP).

Theorem 5. *Let G be a locally compact group. Assume that $PM_p(G)$ has the Radon-Nikodým Property. Then G is discrete. Moreover, G contains no infinite commutative subgroup.*

Proof. If G is separable and $PM_p(G)$ has the RNP, then G is discrete by the main theorem of [4].

Assume that G is nonseparable and nondiscrete. Then G has an open σ-compact subgroup G_0. Since $PM_p(G_0)$ is a closed subspace of $PM_p(G)$, $PM_p(G_0)$ also has the RNP. If G_0 is separable, then G_0 is discrete and hence so is G, which is impossible. Otherwise, there exists a compact normal subgroup K of G_0 with $\lambda(K) = 0$ (where λ is a left Haar measure of G_0) such that G_0/K is separable, but $A_p(G_0/K)$ is isometrically isomorphic to the closed subalgebra of $A_p(G)$ consisting of those functions which are constant on cosets of K. Moreover, this space is complemented in $A_p(G)$. It follows that $PM_p(G_0/K)$ is isometrically isomorphic to a closed subspace of $PM_p(G)$. Therefore $PM_p(G_0/K)$ has the RNP, but then G_0/K is discrete and K is open, which is clearly impossible. It follows that G_0 is discrete and, therefore, so is G.

Let H be a countable commutative subgroup of G. Then $A_p(H)$ is separable and $PM_p(H)$ has the RNP. It follows that $PM_p(H)$ is separable. By Theorem 1, H is finite. Thus G has no infinite commutative subgroup. □

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Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

E-mail address: beforrest@poppy.uwaterloo.ca