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# MULTIPLICATIVE PERTURBATIONS OF LINEAR VOLTERRA EQUATIONS

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ABSTRACT. We prove that the following problems are essentially equivalent:

$$[\mathrm{VO}]_{CT} \qquad u(t) = x + \int_0^t a(t-s)CTu(s)\,ds,$$

$$[\mathrm{VO}]_{TC} \qquad v(t) = y + \int_0^t a(t-s)TCv(s)\,ds,$$

where $T$ is an unbounded closed linear operator in a Banach space $X$ with dense domain $D(T)$, $C$ is a bounded linear operator on $X$, and $a \in L^1_{\mathrm{loc}}([0, \infty), \mathbb{R})$, which is exponentially bounded. We give some applications of our abstract theorem to second-order differential operators on the line.

## 1. INTRODUCTION

The purpose of this note is to study multiplicative perturbations of linear Volterra equations.

Let $X$ be a Banach space and $A$ an unbounded closed linear operator in $X$ with dense domain $D(A)$. Let $a\colon [0, \infty) \to \mathbb{R}$ be a locally integrable function. We suppose that $a(t)$ is Laplace-transformable, i.e., there is $\beta \geq 0$ such that $\int_0^\infty e^{-\beta t}|a(t)|\,dt < \infty$.

We consider the linear Volterra equation

$$[\mathrm{VO}]_A \qquad u(t) = x + \int_0^t a(t-s)Au(s)\,ds \qquad (x \in D(A),\ t \geq 0).$$

Let $(V(t))_{t\geq 0}$ be a family of bounded linear operators in $X$ which is exponentially bounded, i.e., there are constants $M \geq 1$ and $\omega \geq \beta$ such that $\|V(t)\| \leq Me^{\omega t}$ $(t \geq 0)$ is satisfied; $(V(t))_{t\geq 0}$ is said to be of type $(M, \omega)$.

The family $(V(t))_{t\geq 0}$ is called a solution family (or a resolvent) for $[\mathrm{VO}]_A$ if the following conditions are satisfied:

$(V_1)$ $V(t)$ is strongly continuous on $\mathbb{R}_+$, and $V(0) = \mathrm{Id}$.
$(V_2)$ $V(t)$ commutes with $A$, i.e., $V(t)D(A) \subset D(A)$, and $AV(t)x = V(t)Ax$ for all $x \in D(A)$ and $t \geq 0$.

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($V_3$) The following linear Volterra equation holds:

$$V(t)x - x = \int_0^t a(t-s)AV(s)x\,ds \quad \text{for all } x \in D(A) \text{ and } t \geq 0.$$

We can see that if $[\mathrm{VO}]_A$ admits a solution family $(V(t))_{t\geq 0}$ then it is unique and

$$(a * V(t))x := \int_0^t a(t-s)V(s)x\,ds \in D(A)$$

and

$$V(t)x - x = A\int_0^t a(t-s)V(s)x\,ds \quad \text{for all } x \in X \text{ and } t \geq 0$$

(cf. [Pr, §1, Proposition 1.1]).

For the special cases $a(t) \equiv 1$ and $a(t) \equiv t$, the solution family $(V(t))_{t\geq 0}$ for $[\mathrm{VO}]_A$ becomes the $C_0$-semigroup generated by $A$, respectively, the cosine function generated by $A$.

Necessary and sufficient conditions for the existence of a solution family for $[\mathrm{VO}]_A$ have been considered by DaPrato and Iannelli [DaIa], Arendt and Kellermann [ArKe], Prüss [Pr], and others.

We begin with a fundamental theorem of which we will make use later. By $\rho(A)$ we denote the resolvent set of $A$.

**Theorem** (cf. [Pr, §1, Theorem 1.3]). *Assume that $(V(t))_{t\geq 0}$ is strongly continuous and of type $(M, \omega)$. Then $(V(t))_{t\geq 0}$ is a solution family of* $[\mathrm{VO}]_A$ *if and only if the following conditions hold*:

($H_1$) $\hat{a}(\lambda) \neq 0$ *and* $1/\hat{a}(\lambda) \in \rho(A)$ *for all* $\lambda > \omega$.

($H_2$) $H(\lambda, A) := (\lambda - \lambda\hat{a}(\lambda)A)^{-1}$ *exists, and* $H(\lambda, A) = \int_0^\infty e^{\lambda t}V(t)\,dt$ *for all* $\lambda > \omega$.

Let $T$ be a closed linear operator in $X$ with dense domain $D(T)$, and let $C$ be a bounded linear operator on $X$. Consider the operators $TC$ and $CT$ defined by $D(TC) = \{x \in X,\ Cx \in D(T)\}$ and $D(CT) = D(T)$. Our main goal is to show that the perturbation problems $[\mathrm{VO}]_{CT}$ and $[\mathrm{VO}]_{TC}$ are essentially equivalent.

Our result extends the one given by Desch and Schappacher for $C_0$-semigroups (cf. [DeScha2, Theorem 1]). As an application we consider the following ordinary differential operator of second order:

$$(d/dx)^2(c(x)\cdot) + d(x)(d/dx) + e(x).$$

## 2. The main results

Let $X$ be a Banach space. Let $a\colon [0, \infty) \to \mathbb{R}$ be a locally integrable function. We suppose that there exist $\beta \geq 0$ such that

$$\int_0^\infty e^{-\beta t}|a(t)|\,dt < \infty.$$

Let $T$ be an unbounded closed linear operator in $X$ with dense domain $D(T)$, and let $C$ be a bounded linear operator on $X$.

Our main theorem is

**Theorem 2.1.** (a) *If* $[\mathrm{VO}]_{CT}$ *admits a solution family* $(U(t))_{t\geq 0}$ *on* $X$, *then* $[\mathrm{VO}]_{TC}$ *admits a solution family* $(V(t))_{t\geq 0}$ *on* $X$.

(b) *If* $[\mathrm{VO}]_{TC}$ *admits a solution family* $(V(t))_{t\geq 0}$ *on* $X$ *and* $\rho(CT) \neq \varnothing$, *then* $[\mathrm{VO}]_{CT}$ *admits a solution family* $(U(t))_{t\geq 0}$ *on* $X$.

**Corollary 2.2.** *Let* $A$ *be a closed linear operator in* $X$ *with dense domain* $D(A)$ *such that* $[\mathrm{VO}]_A$ *admits a solution family on* $X$. *Let* $B: X_A \to X_A$ *be a bounded linear operator, where* $X_A := (D(A), \|\cdot\|_A)$. *Then* $[\mathrm{VO}]_{A+B}$ *admits a solution family on* $X$.

This corollary is a generalization of a perturbation result given by Desch and Schappacher [DeScha1].

*Proof of Corollary* 2.2. Let $\lambda \in \rho(A)$, and consider $A_\lambda := -\lambda + A + B$. It follows from the proof of Theorem 1.31 [Na] that $\rho(A_\lambda) \neq \varnothing$. We first show that $[\mathrm{VO}]_{A_\lambda}$ admits a solution family on $X$. Since

$$A_\lambda = (I - BR(\lambda, A))(-\lambda + A),$$

it suffices to show that $[\mathrm{VO}]_{B_\lambda}$ admits a solution family on $X$, where $B_\lambda := (-\lambda + A)(I - BR(\lambda, A))$, but this follows from [Rh1, Corollary 1.2] since $B_\lambda = [(-\lambda+A)+(\lambda-A)BR(\lambda, A)]$ and $(\lambda-A)BR(\lambda, A) \in \mathscr{L}(X)$. So again by [Rh1, Corollary 1.2], $[\mathrm{VO}]_{A_\lambda+\lambda}$ admits a solution family on $X$. □

*Proof of Theorem* 2.1. (a) We set

$$V(t)x := x + T\int_0^t a(t-s)U(s)Cx\,ds.$$

Since $\int_0^t a(t-s)U(s)Cx\,ds \in D(CT) = D(T)$, $V(\cdot)x$ is well defined for all $x \in X$. On the other hand, $T$ is closed; hence, $V(t)$ is closed and consequently a bounded linear operator.

Since $[\mathrm{VO}]_{CT}$ admits a solution family, $CT$ is closed, and therefore the graph norms of $CT$ and $T$ are equivalent on $D(T)$ (cf. [Br, Corollary II.6]). Then there exists $\gamma > 0$ such that

$$\|V(t)x\| \leq \|x\| + \gamma\left\|\int_0^t a(t-s)U(s)Cx\,ds\right\|_{CT} \quad \text{for all } x \in X,$$

where $\|\cdot\|_{CT}$ denotes the graph norm of $CT$.

Let $(M, \omega)$ be the type of $(U(t))_{t\geq 0}$. It follows that

$$\begin{aligned}
\|V(t)x\| &\leq \|x\| + M\gamma\left(\int_0^t |a(t-s)|e^{\omega s}\,ds\right)\|Cx\| \\
&\quad + \gamma\left\|CT\int_0^t a(t-s)U(s)Cx\,ds\right\| \\
&\leq \|x\| + M\gamma e^{\omega t}\left(\int_0^t |a(s)|e^{-\omega s}\,ds\right)\|Cx\| + \gamma\|U(t)Cx - Cx\| \\
&\leq \|x\| + M\gamma e^{\omega t}\left(\int_0^\infty |a(s)|e^{-\omega s}\,ds\right)\|Cx\| + \gamma M e^{\omega t}\|Cx\| + \gamma\|Cx\| \\
&\leq M' e^{\omega t}\|x\| \quad \text{for all } x \in X \text{ and some } M' \geq 1,
\end{aligned}$$

so $(V(t))_{t\geq 0}$ is exponentially bounded of type $(M', \omega)$. On the other hand, $V(t)$ is strongly continuous. In fact,

$$\begin{aligned}
\|V(t)x - V(t_0)x\| &= \left\|T\left[\int_0^t a(s)U(t-s)Cx\,ds - \int_0^{t_0} a(s)U(t_0-s)Cx\,ds\right]\right\| \\
&\leq \alpha \left\|\int_0^t a(s)U(t-s)Cx\,ds - \int_0^{t_0} a(s)U(t_0-s)Cx\,ds\right\|_{CT} \\
&= \alpha \left\{\left\|CT\left[\int_0^t a(s)U(t-s)Cx\,ds - \int_0^{t_0} a(s)U(t_0-s)Cx\,ds\right]\right\|\right. \\
&\qquad \left. + \left\|\int_0^t a(s)U(t-s)Cx\,ds - \int_0^{t_0} a(s)U(t_0-s)Cx\,ds\right\|\right\} \\
&= \alpha \left\{\|U(t)Cx - U(t_0)Cx\| + \left\|\int_0^t a(s)U(t-s)Cx\,ds\right.\right. \\
&\qquad \left.\left. - \int_0^{t_0} a(s)U(t_0-s)Cx\,ds\right\|\right\}
\end{aligned}$$

for all $x \in X$ and some $\alpha > 0$.

We only have to show that

$$\int_0^\infty e^{-\lambda t} V(t)\,dt = (\lambda - \lambda\hat{a}(\lambda)TC)^{-1} \quad \text{for all } \lambda > \omega.$$

For $x \in X$ and $\lambda > \omega$,

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} V(t)x\,dt &= \int_0^\infty e^{-\lambda t}\left(x + T\int_0^t a(t-s)U(s)Cx\,ds\right)dt \\
&= \frac{1}{\lambda}x + T\int_0^\infty e^{-\lambda t}\int_0^t a(t-s)U(s)Cx\,ds\,dt.
\end{aligned}$$

Applying the Fubini theorem we obtain

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} V(t)x\,dt &= \frac{1}{\lambda}x + T\int_0^\infty \left(\int_s^\infty e^{-\lambda t}a(t-s)\,dt\right)U(s)Cx\,ds \\
&= \frac{1}{\lambda}x + T\int_0^\infty e^{-\lambda s}\left(\int_0^\infty e^{-\lambda t}a(t)\,dt\right)U(s)Cx\,ds \\
&= \frac{1}{\lambda}x + \frac{\hat{a}(\lambda)}{\lambda}T(I - \hat{a}(\lambda)CT)^{-1}Cx \quad \text{for all } x \in X \text{ and } \lambda > \omega.
\end{aligned}$$

Let $x \in X$, $\lambda > \omega$, and consider $y := x + T(1/\hat{a}(\lambda) - CT)^{-1}Cx$. We will show that $y \in D(TC)$ and $y = (I - \hat{a}(\lambda)TC)^{-1}x$. We have

$$\begin{aligned}
Cy &= Cx + CT\left(\frac{1}{\hat{a}(\lambda)} - CT\right)^{-1}Cx \\
&= Cx + \frac{1}{\hat{a}(\lambda)}\left(\frac{1}{\hat{a}(\lambda)} - CT\right)^{-1}Cx - Cx \\
&= \frac{1}{\hat{a}(\lambda)}\left(\frac{1}{\hat{a}(\lambda)} - CT\right)^{-1}Cx \in D(CT) = D(T);
\end{aligned}$$

thus $y \in D(TC)$ and

$$(I - \hat{a}(\lambda)TC)y = y - \hat{a}(\lambda) \cdot \frac{1}{\hat{a}(\lambda)} T \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx = x.$$

Next, we note that for any $x \in D(TC)$ and $\lambda > \omega$ we have

$$\begin{aligned}
&\left( I + T \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} C \right) (I - \hat{a}(\lambda)TC)x \\
&\quad = x + T \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx - \hat{a}(\lambda) \left( TCx + T \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} CTCx \right) \\
&\quad = x + T \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx - \hat{a}(\lambda) \\
&\qquad \times \left( TCx + T \frac{1}{\hat{a}(\lambda)} \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} Cx - TCx \right) \\
&\quad = x.
\end{aligned}$$

Consequently,

$$I + T \left( \frac{1}{\hat{a}(\lambda)} - CT \right)^{-1} C = (I - \hat{a}(\lambda)TC)^{-1}$$

and

$$\int_0^\infty e^{-\lambda t} V(t)\, dt = \frac{1}{\lambda}(I - \hat{a}(\lambda)TC)^{-1} \quad \text{for } \lambda > \omega.$$

(b) Let $\mu \in \rho(CT)$. We set

$$U(t)x := x + (\mu - CT)C \int_0^t a(t-s)V(s)T(\mu - CT)^{-1} x\, ds.$$

On the other hand, for all $x \in X$,

$$\int_0^t a(t-s)V(s)x\, ds \in D(TC).$$

Hence, $U(\cdot)x$ is well defined for all $x \in X$. Since $T$ is closed, $U(t)$ is a bounded linear operator on $X$. As $t \mapsto TC \int_0^t a(t-s)V(s)x\, ds = V(t)x - x$ and $t \to \int_0^t a(s)V(t-s)x\, ds$ are continuous, we conclude that $U(t)$ is strongly continuous. On the other hand, it is clear that $U(\cdot)$ is exponentially bounded of type $(M, \omega)$, where $M \geq M'$ $((M', \omega)$ is the type of $(V(t))_{t \geq 0})$; therefore, it suffices to show that

$$\int_0^\infty e^{-\lambda t} U(t)\, dt = (\lambda - \lambda \hat{a}(\lambda)CT)^{-1}$$

for all $\lambda > \omega$.

For $x \in X$ and $\lambda > \omega$,

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t} U(t)x\,dt \\
&\quad = \frac{1}{\lambda}x + \int_0^\infty e^{-\lambda t}(\mu - CT)C \int_0^t a(t-s)V(s)T(\mu - CT)^{-1}x\,ds\,dt \\
&\quad = \frac{1}{\lambda}x + (\mu - CT)C \int_0^\infty e^{-\lambda t} \int_0^t a(t-s)V(s)T(\mu - CT)^{-1}x\,ds\,dt \\
&\quad = \frac{1}{\lambda}x + (\mu - CT)C \int_0^\infty \left(\int_s^\infty e^{-\lambda t}a(t-s)\,dt\right) V(s)T(\mu - CT)^{-1}x\,ds \\
&\quad = \frac{1}{\lambda}x + (\mu - CT)C \int_0^\infty \left(\int_0^\infty e^{-\lambda t}a(t)\,dt\right) e^{-\lambda s}V(s)T(\mu - CT)^{-1}x\,ds \\
&\quad = \frac{1}{\lambda}x + \hat{a}(\lambda)(\mu - CT)C(I - \hat{a}(\lambda)TC)^{-1}T(\mu - CT)^{-1}x \\
&\quad = \frac{1}{\lambda}\left[I + (\mu - CT)C\left(\frac{1}{\hat{a}(\lambda)} - TC\right)^{-1} T(\mu - CT)^{-1}\right]x.
\end{aligned}$$

For $x \in X$ and $\lambda > \omega$ we put

$$y := x + (\mu - CT)C\left(\frac{1}{\hat{a}(\lambda)} - TC\right)^{-1} T(\mu - CT)^{-1}x.$$

We will show that $y \in D(CT)$ and $(I - \hat{a}(\lambda)CT)y = x$. An elementary calculation gives

$$\begin{aligned}
y = &\left(\mu - \frac{1}{\hat{a}(\lambda)}\right) C\left(\frac{1}{\hat{a}(\lambda)} - TC\right)^{-1} T(\mu - CT)^{-1}x + \mu(\mu - CT)^{-1}x \\
&\in D(CT) = D(T)
\end{aligned}$$

and

$$\begin{aligned}
\hat{a}(\lambda)CTy = \hat{a}(\lambda)&\left(\mu - \frac{1}{\hat{a}(\lambda)}\right) \\
&\times C\left[\frac{1}{\hat{a}(\lambda)}\left(\frac{1}{\hat{a}(\lambda)} - TC\right)^{-1} T(\mu - CT)^{-1}x - T(\mu - CT)^{-1}x\right] \\
&+ \mu\hat{a}(\lambda)CT(\mu - CT)^{-1}x \\
= y - x&.
\end{aligned}$$

Consequently, $(I - \hat{a}(\lambda)CT)y = x$.

On the other hand, we have, for any $x \in D(T)$,

$$\begin{aligned}
&\left[I + (\mu - CT)C\left(\frac{1}{\hat{a}(\lambda)} - TC\right)^{-1} T(\mu - CT)^{-1}\right] CTx \\
&\qquad = CT\left[I + (\mu - CT)C\left(\frac{1}{\hat{a}(\lambda)} - TC\right)^{-1} T(\mu - CT)^{-1}\right] x;
\end{aligned}$$

hence, $I + (\mu - CT)C(1/\hat{a}(\lambda) - TC)^{-1}T(\mu - CT)^{-1} = (I - \hat{a}(\lambda)CT)^{-1}$. □

## 3. Applications

In this section we describe two applications of Theorem 2.1 to the following ordinary differential operator of second order

$$\left(\frac{d}{dx}\right)^2 (c(x)\cdot) + d(x)\left(\frac{d}{dx}\right) + e(x)$$

with $a(t) \equiv t$.

**Example 3.1.** Let $I$ be a bounded and closed subinterval of $\mathbb{R}$, and consider $X = C(I)$, the Banach space of continuous functions in $x \in I$ with norm $\|u\|_\infty = \max_{x\in I} |u(x)|$.

Assume

(i) $c(x) > 0$ for all $x \in I$,

(ii) $c(\cdot)$, $c'(\cdot)$, $d(\cdot)$, and $e(\cdot)$ belong to $C(I)$.

Consider the operator $T$ in $X$ defined by

$$Tu = u'' + d(x)u' + e(x)u,$$
$$D(T) = \{u \in X;\ u'' \in X;\ u'(x) = 0 \text{ for } x \in \partial I\},$$

and let $C\colon X \to X$ be given by $Cu = c \cdot u$.

It is known that $CT$ generates a cosine function on $X$ (cf. [WaSe, §3]); hence, by Theorem 2.1, $TC$ also generates a cosine function on $X$. The above argument gives an operator-theoretical approach to the following initial boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}(c\cdot u) + d(x)\frac{\partial}{\partial x}(c\cdot u) + e(x)c(x)\cdot u, \qquad t \in \mathbb{R},\ x \in I,$$
$$u(0, x) = f(x), \quad \frac{\partial}{\partial t}u(0, x) = g(x), \qquad x \in I\ (f, g \in D(TC)). \quad \square$$

**Example 3.2.** Let $X = L^2(I)$, where $I = (0, 1)$, and define $T$ on $X$ by $Tu = u''$ with $D(T) = \{f \in H^2(0, 1): f(0) = f(1) = 0\}$ (Dirichlet boundary conditions) or $D(T) = \{f \in H^2(0, 1): f'(0) = f'(1) = 0\}$ (Neumann boundary conditions). Then $T$ is selfadjoint and form-negative, so $T$ generates a cosine function on $X$ (cf. [Fa, Theorems 2.2 and 5.1]).

Now let $c \in W^{1,\infty}(I)$ such that $c(x) \geq \delta$ for all $x \in I$ and some constant $\delta > 0$.

We consider the operator $C$ defined by $Cu = c \cdot u$. Then $CT$ generates a cosine function on $X$. In fact, the function $\varphi(x) = \int_0^x 1/\sqrt{c(s)}\,ds$ is a homeomorphism of $I$ onto another interval $J$. Moreover, $\varphi$ induces an isomorphism between $L^2(I)$ and $L^2(J)$ defined by $V\colon L^2(J) \to L^2(I)$, $Vf := f \circ \varphi$.

For $u \in D(CT) = D(T)$ we have

$$(V^{-1}CTV)u = (c \circ \varphi^{-1} \cdot \varphi'' \circ \varphi^{-1})u' + u'' := b(\cdot)u' + u'',$$

where $b(\cdot) := c \circ \varphi^{-1}(\cdot) \cdot \varphi'' \circ \varphi^{-1}(\cdot)$. On the other hand, it follows from [Fa, Theorem 2.2, p. 105 and Theorem 5.1, p. 116] that

$$E := \{x \in L^2(J): U(t)x \text{ is continuously differentiable in } t \in \mathbb{R}\}$$
$$= \begin{cases} H_0^1(J) & \text{in the case of Dirichlet boundary conditions,} \\ H^1(J) & \text{in the case of Neumann boundary conditions,} \end{cases}$$

where $(U(t))_{t\geq 0}$ is a cosine function on $L^2(J)$ with generator $\widetilde{T}$ defined by $\widetilde{T}u = u''$ $(u \in L^2(J))$. It follows from the Kisynski theorem (cf. [Ki] or [Wa]) that the matrix operator

$$\mathscr{T} := \begin{pmatrix} 0 & I \\ \widetilde{T} & 0 \end{pmatrix} \quad \text{with domain } D(\widetilde{T}) \times E$$

is the infinitesimal generator of a strongly continuous group on $E \times L^2(J)$. Since $\mathscr{B} \in \mathscr{L}(E \times L^2(J))$, $\mathscr{B} + \mathscr{T}$ generates a strongly continuous group in $E \times L^2(J)$, where

$$\mathscr{B} := \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \qquad (Bu = b(\cdot)u' \text{ for } u \in E).$$

By [Wa, Theorem], $(V^{-1}CTV)$ generates a cosine function on $L^2(J)$. Consequently, $CT$ generates a cosine function on $X$. Applying Theorem 2.1, we get that $TC$ generates a cosine function on $X$, i.e., the following problem is well posed:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}(c \cdot u), \qquad t \in \mathbb{R},\ x \in I,$$

$$u(0, x) = f(x), \quad \frac{\partial}{\partial t}u(0, x) = g(x), \qquad x \in I\ (f, g \in D(TC)). \quad \square$$

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