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COUNTABLE PRODUCTIVITY OF A CLASS OF PSEUDORADIAL SPACES

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ABSTRACT. It is shown that the class of compact R-monolithic spaces is countably productive.

0. INTRODUCTION

A basic problem in the theory of pseudoradial (or chain-net) spaces is the behaviour of this class of spaces under the product operation. Recently Juhász and Szentmiklossy [6] proved that assuming $2^\omega \le \omega_2$ the product of countably many compact pseudoradial spaces is pseudoradial. In ZFC, it is unknown whether even the product of two compact pseudoradial spaces is pseudoradial. Observe, however, that the productivity of pseudoradiality, in general, can be expected to hold only in the class of compact spaces. For instance, $I \times l(\omega_1)$, where I is the unit segment and $l(\omega_1)$ is the one point Lindelofization of a discrete space of cardinality ω_1, is a product of two "very good" pseudoradial spaces (one compact metric and the other radial and Lindelöf) that is not pseudoradial.

Some positive results have been found for certain classes of spaces. For example, in [5] it is shown that the product of a compact radial space and a compact pseudoradial space is pseudoradial, and in [3] it is shown that the product of two compact pseudoradial spaces, one of which is R-monolithic, is pseudoradial. Later both of these results have been simultaneously generalized in [4].

In this paper we prove that the class of compact R-monolithic spaces is countably productive. In particular, this result improves an analogous one stated in [1], where it is proved that a countable product of compact biradial spaces is pseudoradial.

1. BASIC CONCEPTS

Henceforth the term sequence means transfinite sequence, i.e., a set of the form $\{x_\alpha : \alpha \in \kappa\}$. A sequence $\{x_\alpha : \alpha \in \kappa\}$ of points in the topological space X is said to be strictly convergent to the point x if it converges to x, κ is

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regular, and $x \notin \overline{\{x_\alpha : \alpha \in \beta\}}$ for every $\beta \in \kappa$. A subset A of X is said to be chain (respectively, strictly chain) closed if it contains the limit points of all convergent (respectively, strictly convergent) sequences contained in it.

A topological space X is pseudoradial (almost radial) if every chain (strictly chain) closed subset is closed. X is radial if the closure of any subset can be obtained by adjoining to it the limit points of all the convergent sequences contained in it.

The chain character of a pseudoradial space X, denoted by $\sigma_c(X)$, is the smallest cardinal κ such that for any nonclosed set $A \subset X$ there exists a point $x \in \overline{A} \setminus A$ and a sequence $\{x_\alpha : \alpha \in \lambda\}$ which converges to x and satisfies $\lambda \leq \kappa$.

We say that a pseudoradial space X is R-monolithic if for any $A \subset X$ the inequality $\sigma_c(\overline{A}) \leq |A|$ holds.

Every compact monolithic, or more generally quasi-monolithic (see [1]), space is R-monolithic.

Sequential $\rightarrow$ almost radial $\rightarrow$ pseudoradial.

Frechet-Urysohn $\rightarrow$ radial $\rightarrow$ almost radial.

A compact sequential non-Frechet-Urysohn space and the space $\delta(N)$ in [8, Example 4.83] shows that R-monolithic $\not\rightarrow$ radial and radial $\not\rightarrow$ R-monolithic, even in the class of compact spaces. For more information on pseudoradial and related spaces see [9].

A subset A of a topological space X is said to be κ-closed ($< \kappa$-closed) if $\overline{B} \subset A$ whenever $B \subset A$ and $|B| \leq \kappa$ ($|B| < \kappa$). The tightness of X, denoted by $t(X)$, is the smallest cardinal κ such that every κ-closed subset of X is closed.

2. Results

Theorem 1. *The class of R-monolithic compact spaces is countably productive.*

Proof. We begin by proving that the class of R-monolithic compact spaces is finitely productive. To this end it is enough to show that the product of two R-monolithic compact spaces, say X and Y, is R-monolithic. In fact, it suffices to show that $X \times Y$ is almost radial. The reason is that for any subset A of $X \times Y$ we have $\overline{A} \subset \overline{\pi_X(A)} \times \overline{\pi_Y(A)}$ and consequently (see [7]) $t(\overline{A}) \leq t(\overline{\pi_X(A)})t(\overline{\pi_Y(A)}) \leq \sigma_c(\overline{\pi_X(A)})\sigma_c(\overline{\pi_Y(A)}) \leq |A|$. Thus, recalling (see [2]) that in every almost radial space the tightness equals the chain character, we get $\sigma_c(\overline{A}) \leq |A|$. Assume that $X \times Y$ is not almost radial, and let κ be the smallest cardinal such that there exists a strictly chain closed set $A \subset X \times Y$ which is not κ-closed. Select a set $B \subset A$ satisfying $|B| \leq \kappa$ and $\overline{B} \setminus A \neq \varnothing$, and fix a point $(x, y) \in \overline{B} \setminus A$. Since an R-monolithic space is almost radial and $\{x\} \times Y \cap A$ is strictly chain closed, it follows that $\{x\} \times Y \cap A$ is actually closed. Therefore, by replacing A with a convenient subset, we can assume that $\pi_X(x) \notin \pi_X(A)$. This means that $\overline{\pi_X(B)} \setminus \pi_X(A) \neq \varnothing$. By hypothesis $\sigma_c(\overline{\pi_X(B)}) \leq |B|$ and so, taking also into account that $\pi_X(A)$ is $< \kappa$-closed, there exists a sequence $\{x_\alpha : \alpha \in \kappa\} \subset \pi_X(A)$ converging to a point $x' \notin \pi_X(A)$. For any α pick a point y_α in such a way that $(x_\alpha, y_\alpha) \in A$, and let z be a complete accumulation point of the set $\{y_\alpha : \alpha \in \kappa\}$. $X \times \{z\} \cap A$ is a closed set missing (x', z), and consequently, by replacing A with a convenient closed subset, we can assume that $z \notin \pi_Y(A)$. Letting $C_\alpha = \overline{\{y_\beta : \beta \in \alpha\}}$ and

$C = \bigcup_{\alpha\in\kappa} C_\alpha$, we see that C is not closed but $< \kappa$-closed. Thus, recalling that $\sigma_c(\overline{C}) \leq \kappa$, there exists a sequence $\{y'_\alpha : \alpha \in \kappa\} \subset C$ converging to a point $y' \notin C$. Obviously we can assume that $y'_\alpha \notin C_\alpha$ and a function $f: \kappa \to \kappa$ can be defined in such a way that $y'_\alpha \in \overline{\{y_\beta : \alpha \in \beta \in f(\alpha)\}}$. Now, picking a point $x'_\alpha \in \overline{\{x_\beta : \alpha \in \beta \in f(\alpha)\}}$ so that $(x'_\alpha, y'_\alpha) \in A$, we obtain a sequence in A strictly converging to $(x', y') \notin A$, which is a contradiction. The argument used in this paragraph will be needed again later and we refer to it as $(*)$.

Now we proceed with the proof of the theorem. Let $\{X_n : n \in \omega\}$ be a family of compact R-monolithic spaces, and put $X = \prod_{n\in\omega} X_n$. Indicate with π_n the projection of X onto the product of the first $n+1$ factors. Tightness is preserved in a product of countably many compact spaces (see [7]) and hence, by the same reasoning exhibited at the beginning of the present proof, we have only to show that X is almost radial. By contradiction, assume that there exists a nonclosed strictly chain closed subset A of X and let κ be the smallest cardinal such that A is not κ-closed. Let B be a subset of A such that $|B| = \kappa$ and $\overline{B}\backslash A \neq \varnothing$, and choose a point $x \in \overline{B}\backslash A$. Suppose that $\pi_n(x) \in \pi_n(A)$ for every n, and choose points $x_n \in A$ in such a way that $\pi_n(x_n) = \pi_n(x)$ for every n. It is easy to see that the sequence $\{x_n : n \in \omega\}$ converges to x, in contrast with the chain closedness of A. Thus there must exist some integer m so that $\pi_m(x) \notin \pi_m(A)$ and consequently $\pi_m(A)$ is not closed in $\pi_n(X)$. Thanks to the finite productivity of R-monolithicity, we can assume, without any loss of generality, that $m = 0$. It is clear that $\pi_0(A)$ is not κ-closed but $< \kappa$-closed and so there exists a sequence $\{x_{0,\alpha} : \alpha \in \kappa\} \subset A$ such that the sequence $\{\pi_0(x_{0,\alpha}) : \alpha \in \kappa\}$ converges to a point $x'_0 \in X_0\backslash\pi_0(A)$. Obviously κ must be regular. Now, arguing as in the construction $(*)$, we can define for every $n \in \omega$ a sequence $\{x_{n,\alpha} : \alpha \in \kappa\} \subset A$ in such a way that the sequence $\{\pi_n(x_{n,\alpha}) : \alpha \in \kappa\}$ converges to the point $(x'_0, x'_1, \dots, x'_n)$. Let x' be the point of X whose nth coordinate is x'_n. If $\kappa = \omega$ then a quick look to the construction $(*)$ shows that the sequence $\{x_{n+1,m} : m \in \omega\}$ can be taken as a subsequence of $\{x_{n,m} : m \in \omega\}$. The diagonal sequence $\{x_{n,n} : n \in \omega\}$ converges to x', in contradiction with the chain closedness of A. Therefore, κ must be uncountable. For every α select an accumulation point x'_α of the set $\{x_{n,\alpha} : n \in \omega\}$. As A is ω-closed, the point x'_α belongs to A for every α. The sequence $\{x'_\alpha : \alpha \in \kappa\}$ converges to x'. To check this, it is sufficient to consider neighbourhoods of x' of the form $\pi_m^{-1}(U)$, where U is a closed neighbourhood of $\pi_m(x')$ in $\pi_m(X)$. By construction, for every $n \geq m$ there exists an ordinal $\alpha_n \in \kappa$ such that $\pi_n(x_{n,\alpha}) \in U \times X_{m+1} \times \cdots \times X_n$ whenever $\alpha \geq \alpha_n$. Let $\hat{\alpha} = \sup\{\alpha_n : n \geq m\}$. For any $\alpha \geq \hat{\alpha}$ and any $n \geq m$ we have $x_{n,\alpha} \in \pi_m^{-1}(U)$ and, therefore, $x'_\alpha \in \pi_m^{-1}(U)$. This shows the convergence (obviously strict) of $\{x'_\alpha : \alpha \in \kappa\}$ to $x' \in \overline{A}\backslash A$. Again we reach a contradiction and the proof is complete.

In [6] it is shown that a compact sequentially compact space X is pseudoradial provided that every closed subspace of X having density less than 2^ω is pseudoradial. Taking this into account and recalling that a pseudoradial compact space is always sequentially compact, we see that the product of countably many compact pseudoradial spaces is pseudoradial if all the "small" closed subspaces of them are R-monolithic.

Theorem 2. *The class of compact pseudoradial spaces whose closed subspaces of density less than the continuum are R-monolithic is countably productive.*

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