

ON THE RIEMANNIAN GEOMETRY OF THE NILPOTENT GROUPS $H(p, r)$

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ABSTRACT. We study some aspect of the left-invariant Riemannian geometry on a class of nilpotent Lie groups $H(p, r)$ that generalize the Heisenberg group H_{2p+1} . Let us prove that the groups of type H (or Kaplan's spaces) and the $H(p, r)$ groups have same common Riemannian properties but they are not the same spaces.

INTRODUCTION

The Heisenberg group H_{2p+1} with the left-invariant metric

$$ds^2 = (dx_i)^2 + \left(dz + \sum x_{2i-1} dx_{2i}\right)^2$$

is a typical model of a homogeneous Riemannian non-Euclidean structure.

The geometry of these metrics is strongly connected to contact geometry of the Pfaff equation

$$\omega = dz + \sum x_{2i-1} dx_{2i} = 0.$$

In fact, let $\mathcal{C}(\omega)$ be the group of contact transformations relative to ω (i.e., of the transformations preserving the codimension 1 distribution $\text{Ker}(\omega)$). Then

$$\mathcal{C}(\omega) = \mathcal{I}som(ds^2),$$

where $\mathcal{I}som(ds^2)$ denotes the group of isometries of ds^2 .

It is natural to study the Riemannian structures adapted to a generalized (i.e., of higher codimension) contact geometry.

Recall that in codimension 1, every contact equation is equivalent to $\omega = dz + \sum x_{2i-1} dx_{2i} = 0$. This is not true anymore in codimension greater than 1, where one has an infinity of models $[G_1]$.

In [GH] Haraguchi and the second author introduced a notion of *r-contact system* that seems to generalize in a remarkable way that of codimension 1 contact structure.

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Definition. Let S be a Pfaff system. S is called an r -contact system if the rank of S is r , its class is maximal, and S admits an integral foliation of dimension $(n-r)/(r+1)$ (which is the maximal dimension of such a foliation).

Theorem [H]. Let M be an $(rp+r+p)$ -dimensional manifold with an r -contact system S . Then S is locally equivalent to the system given by

$$\omega_i = dz_i + \sum_{\alpha=1}^p y_i^\alpha dx_\alpha, \quad i = 1, \dots, r.$$

The simplest examples of Lie groups admitting a left-invariant r -contact system are groups that generalize the Heisenberg group H_{2p+1} , denoted by $H(p, r)$ (see [GH]).

In this paper, we study some aspect of the Riemannian geometry of $H(p, r)$ equipped with a natural left-invariant metric whose isometries preserve the distribution associated to the r -contact system.

I. THE GROUPS $H(p, r)$

1.1. Lie algebra considerations.

Definition 1.1. A generalized Heisenberg group in the sense of Goze and Haraguchi [GH] is the product $H(p, r) = \mathcal{M}_{1p} \times \mathcal{M}_{pr} \times \mathcal{M}_{1r}$ of three Abelian topological groups of matrices of dimensions $1 \times p$, $p \times r$, and $1 \times r$ respectively, endowed with the multiplication

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)).$$

In [H] and [GH] Haraguchi and Goze proved the following result:

Proposition 1.2. (1) *The groups $H(p, r)$ are $(rp+r+p)$ -dimensional, two-step nilpotent, connected, and simply connected.*

(2) *The center Z of $H(p, r)$ is r -dimensional and isomorphic with the Abelian topological group \mathcal{M}_{1r} .*

(3) *A group $H(p, r)$ admits discrete uniform subgroups. An attempt at classification can be found in [H].*

(4) *A group $H(p, r)$ admits an r -contact system (see Introduction).*

We shall use on $H(p, r)$ the global left-invariant coframe

$$(1.1) \quad \begin{cases} \vartheta^\alpha = dx_\alpha, \vartheta^{(\alpha, i)} = dy_i^\alpha, & i = 1, \dots, r; \\ \vartheta^i = dz_i + \frac{1}{2}(y_i^\alpha dx_\alpha - x_\alpha dy_i^\alpha), & \alpha = 1, \dots, p. \end{cases}$$

The frame of left-invariant vector fields dual to the 1-forms (1.1) is

$$E_\alpha = \frac{\partial}{\partial x_\alpha} - \frac{1}{2}y_i^\alpha \frac{\partial}{\partial z_i}, \quad E_{(\alpha, i)} = \frac{\partial}{\partial y_i^\alpha} + \frac{1}{2}x_\alpha \frac{\partial}{\partial z_i}, \quad E_i = \frac{\partial}{\partial z_i}.$$

The Maurer-Cartan equations for the Lie algebra $\mathfrak{h}(p, r)$ are given by

$$\begin{cases} d\vartheta^\alpha = 0, \\ d\vartheta^{(\alpha, i)} = 0, \\ d\vartheta^i = -\vartheta^\alpha \wedge \vartheta^{(\alpha, i)}. \end{cases}$$

Remarks 1.3. From the Maurer-Cartan equations it follows at once that:

(1) the group $H(p, r)$ is isomorphic to the Heisenberg group H_{2p+1} if and only if $\dim Z = r = 1$;

(2) the derived group $H'(p, r)$ is an r -dimensional group and coincides with the center Z of $H(p, r)$;

(3) the Lie algebra $\mathfrak{h}(p, r)$ of $H(p, r)$ is the direct sum of three Abelian subalgebras $\mathfrak{h}(p, r) = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathcal{Z}$, where \mathcal{Z} is the center of $\mathfrak{h}(p, r)$ (the Lie algebra of the center Z of $H(p, r)$).

1.2. A Riemannian structure for $H(p, r)$. We shall consider on the group $H(p, r)$ the left-invariant metric tensor g given by

$$(1.2) \quad g = \sum (\vartheta^\alpha)^2 + \sum (\vartheta^{(\alpha, i)})^2 + \sum (\vartheta^i)^2.$$

Therefore, with respect to the metric g , the vector fields $\{E_A\}$ form an orthonormal frame.

To obtain the Levi-Civita connection we compute the connection 1-forms (by means of $d\vartheta^A = \vartheta^B \wedge \vartheta_B^A$)

$$\vartheta_\alpha^i = -\frac{1}{2}\vartheta^{(\alpha, i)}, \quad \vartheta_{(\alpha, i)}^\alpha = \frac{1}{2}\vartheta^\alpha, \quad \vartheta_{(\alpha, i)}^\beta = \frac{1}{2}\vartheta^\beta.$$

The curvature forms ($\Omega_B^A = d\vartheta_B^A + \vartheta_B^C \wedge \vartheta_C^A$) are

$$\begin{aligned} \Omega_j^i &= \frac{1}{4}\vartheta^{(\alpha, j)} \wedge \vartheta^{(\alpha, i)}, & \Omega_\beta^\alpha &= \frac{1}{4}\vartheta^{(\alpha, i)} \wedge \vartheta^{(\beta, i)}, \\ \Omega_{(\alpha, j)}^{(\alpha, i)} &= \frac{1}{4}\vartheta^j \wedge \vartheta^i, & \Omega_{(\beta, i)}^{(\alpha, i)} &= \frac{1}{4}\vartheta^\beta \wedge \vartheta^\alpha, \\ \Omega_{(\beta, j)}^{(\alpha, i)} &= 0, & \Omega_{(\alpha, j)}^i &= \frac{1}{4}\vartheta^j \wedge \vartheta^{(\alpha, i)}, \\ \Omega_{(\alpha, i)}^i &= \frac{1}{4}\vartheta^i \wedge \vartheta^{(\alpha, i)}, & \Omega_\alpha^i &= \frac{1}{4}\vartheta^i \wedge \vartheta^\alpha, \\ \Omega_{(\alpha, i)}^\alpha &= -\frac{3}{4}\vartheta^\alpha \wedge \vartheta^{(\alpha, i)}, & \Omega_{(\alpha, i)}^\beta &= -\frac{1}{4}\vartheta^\beta \wedge \vartheta^{(\alpha, i)}. \end{aligned}$$

Hence, the Ricci tensor ρ is given by

$$(1.3) \quad \begin{cases} \rho(E_i, E_j) = \frac{r}{2}\delta_j^i, & i, j = 1, \dots, r; \\ \rho(E_\alpha, E_\beta) = -\frac{r}{2}\delta_\beta^\alpha, & \alpha, \beta = 1, \dots, p; \\ \rho(E_{(\alpha, i)}, E_{(\beta, j)}) = -\frac{1}{2}\delta_j^i\delta_\beta^\alpha. \end{cases}$$

As a consequence [J], the Riemannian space $(H(p, r), g)$ is not an Einstein space (i.e., $\rho_{AB} \neq K g_{AB}$, $A, B = 1, \dots, rp + r + p$).

Finally we compute the scalar curvature, which is $\tau = \sum \rho_{AB} = -\frac{1}{2}rp$.

1.3. Geodesics and Killing vector fields on $(H(p, r), g)$. Further, let $J: \mathcal{Z} \rightarrow \text{End}(\mathfrak{h}_1 \oplus \mathfrak{h}_2)$ be the linear map defined by

$$g(J(a)X, Y) = g([X, Y], a),$$

where $a \in \mathcal{Z}$ and $X, Y \in \mathfrak{h}_1 \oplus \mathfrak{h}_2$. It is easy to see that the endomorphism $J(a)$ satisfies the following conditions:

$$(1.4) \quad \begin{aligned} \|J(a)X\| &= \|X\| \|a\|, & a \in \mathcal{Z}, X \in \mathfrak{h}_1 \oplus \mathfrak{h}_2; \\ J(a)^2 &= -\|a\|^2 I; & g(J(a)X, Y) + g(J(a)Y, X) &= 0. \end{aligned}$$

Using polarization we obtain from (1.4)

$$\begin{aligned} g(J(a)X, J(b)X) &= g(a, b)\|X\|^2, \\ g(J(a)X, J(a)Y) &= \|a\|^2 g(X, Y) \end{aligned}$$

for all $X, Y \in \mathfrak{h}_1 \oplus \mathfrak{h}_2$ and $a, b \in \mathcal{Z}$.

The geodesics of $(H(p, r), g)$ are the solutions of Euler-Lagrange equations for the Lagrangian

$$L = \frac{1}{2}[(\dot{x}_\alpha)^2 + (\dot{y}_i^\alpha)^2 + (\dot{z}_i + \frac{1}{2}y_i^\alpha \dot{x}_\alpha - \frac{1}{2}x_\alpha \dot{y}_i^\alpha)^2]$$

associated to the metric (1.2). These equations are

$$\begin{aligned} (1.5) \quad \ddot{x}_\alpha &= -\dot{y}_i^\alpha (\dot{z}_i + \frac{1}{2}y_i^\alpha \dot{x}_\alpha - \frac{1}{2}x_\alpha \dot{y}_i^\alpha), \\ \ddot{y}_i^\alpha &= \dot{x}_i^\alpha (\dot{z}_i + \frac{1}{2}y_i^\alpha \dot{x}_\alpha - \frac{1}{2}x_\alpha \dot{y}_i^\alpha), \\ \frac{d}{dt}(\dot{z}_i + \frac{1}{2}y_i^\alpha \dot{x}_\alpha - \frac{1}{2}x_\alpha \dot{y}_i^\alpha) &= 0. \end{aligned}$$

The last equation implies at once $\dot{z}_i + \frac{1}{2}y_i^\alpha \dot{x}_\alpha - \frac{1}{2}x_\alpha \dot{y}_i^\alpha = \text{const}$. We restrict our attention to the geodesics $\gamma(t) = (x(t), y(t), z(t))$, starting at identity with the velocity vector $\dot{\gamma}(0) = (\lambda, \mu, \nu)$, i.e., satisfying the initial condition $x(0) = y(0) = z(0) = 0$, and $\dot{x}(0) = \lambda$, $\dot{y}(0) = \mu$, $\dot{z}(0) = \nu$. Then the last of (1.5) becomes

$$\dot{z}_i(t) + \frac{1}{2}y_i^\alpha(t)\dot{x}_\alpha(t) - \frac{1}{2}x_\alpha(t)\dot{y}_i^\alpha(t) = \nu_i \quad \text{for all } t,$$

and the first two equations in (1.5) reduce to

$$\begin{cases} \ddot{x}_\alpha = -\nu_i \dot{y}_i^\alpha, \\ \ddot{y}_i^\alpha = \nu_i \dot{x}_\alpha. \end{cases}$$

Then the equations of these geodesics are, if $\nu \neq 0$,

$$\begin{cases} x(t) = \frac{1 - \cos(\|\nu\|t)}{\|\nu\|^2} J(\nu)\mu + \frac{\lambda}{\|\nu\|} \sin(\|\nu\|t), \\ y(t) = \frac{1 - \cos(\|\nu\|t)}{\|\nu\|^2} J(\nu)\lambda - \frac{H}{\|\nu\|} \sin(\|\nu\|t) + (H + \mu)t, \\ z(t) = \frac{1 - \cos(\|\nu\|t)}{\|\nu\|} L - \frac{N}{\|\nu\|} \sin(\|\nu\|t) + (1 + K\nu - 2M \cos(\|\nu\|t))t, \end{cases}$$

where

$$\begin{aligned} H_{(\alpha, i)} &= \frac{\mu_{(\alpha, j)} \nu_j \nu_i}{\|\nu\|^2}; \quad L_i = H_{(\alpha, i)} \lambda_\alpha + \mu_{(\alpha, i)} \lambda_\alpha; \quad M_i = \frac{H_{(\alpha, i)} \mu_{(\alpha, j)} \nu_j}{\|\nu\|^2}; \\ K &= \frac{\|\lambda\|^2 + \|\mu\|^2 + 2\|J(\nu)\mu\|^2}{\|\nu\|^2}; \quad N = M + \frac{Lt}{2} + K\nu; \end{aligned}$$

and

$$\begin{cases} x(t) = \lambda t, \\ y(t) = \mu t, \\ z(t) = 0. \end{cases} \quad \text{if } \nu = 0,$$

Now recall that $J(\nu)^2 = -\|\nu\|^2 \text{Id}$. Then if $H + \mu = 0$, except when $\nu = 0$, for a geodesic γ starting at the identity with initial vector (λ, μ, ν) , one obtains

the expression

$$(1.6) \quad \begin{aligned} X(t) &= \frac{(I - e^{tJ(\nu)})}{\|\nu\|^2} J(\nu)\xi, \\ Z(t) &= \left(t + \frac{1}{2} \frac{\|\xi\|^2}{\|\nu\|^2} \left(t - \frac{\sin\|\xi\|t}{\|\xi\|} \right) \right) \nu, \end{aligned}$$

where $X(t) = (x(t), y(t))$, $Z(t) = z(t)$, and $\xi = \lambda + \mu$.

Proposition 1.4. Let \mathcal{I}_g be the group of isometries of $(H(p, r), g)$.

- (i) If $r > 1$ then $\dim \mathcal{I}_g = rp + p + r + p(p-1)/2 + r(r-1)/2$.
- (ii) If $r = 1$ then $\dim \mathcal{I}_g = (p+1)^2$.

Proof. It is sufficient to compute the Killing vector fields. For $r > 1$ a basis of Killing vector fields on $(H(p, r), g)$ has been found in [R]. It is

$$\begin{aligned} \frac{\partial}{\partial x_\alpha} + y_i^\alpha \frac{\partial}{\partial z_i}, \quad \frac{\partial}{\partial y_i^\alpha} - x_\alpha \frac{\partial}{\partial z_i}, \quad \frac{\partial}{\partial z_i}, \\ y_i^\beta \frac{\partial}{\partial y_i^\alpha} - y_i^\alpha \frac{\partial}{\partial y_i^\beta} - x_\beta \frac{\partial}{\partial x_\alpha} + x_\alpha \frac{\partial}{\partial x_\beta}, \\ y_i^\alpha \frac{\partial}{\partial y_j^\alpha} - y_j^\alpha \frac{\partial}{\partial y_i^\alpha} - z_j \frac{\partial}{\partial z_i} + z_i \frac{\partial}{\partial z_j}. \end{aligned}$$

For $r = 1$ a basis is

$$\begin{aligned} \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial z}, \\ y_i \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial y_i} - x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}, \\ x_i \frac{\partial}{\partial y_j} - x_j \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial x_i}. \end{aligned}$$

II. NATURAL REDUCTIVITY

II.1. Definition of a Riemannian homogeneous naturally reductive space. Let (M, g) be a connected n -dimensional Riemannian homogeneous manifold. Further let $M = G/K$, where G is a group of isometries for M and K is the isotropy subgroup at a point p of M .

We denote by \mathfrak{g} (respectively \mathfrak{k}) the Lie algebra of G (respectively K). Then the space $M = G/K$ is called *naturally reductive* if there exists a vector subspace \mathfrak{m} of \mathfrak{g} such that

$$\begin{aligned} \mathfrak{g} = \mathfrak{m} \oplus \mathfrak{k}; \quad [\mathfrak{m}, \mathfrak{k}] \subseteq \mathfrak{m}; \\ \langle [X, Y]_{\mathfrak{m}}, Z \rangle + \langle [X, Z]_{\mathfrak{m}}, Y \rangle = 0, \quad X, Y, Z \in \mathfrak{m}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product induced on \mathfrak{m} from g by identification of \mathfrak{m} with $T_p M$. Further $[\cdot]_{\mathfrak{m}}$ is the projection of $[\cdot]$ on \mathfrak{m} .

In [TV] Tricerri and Vanhecke proved the following result:

Theorem 2.1. *Let (M, g) be a connected, simply connected, and complete Riemannian manifold. Then (M, g) is a naturally reductive homogeneous space if and only if there exists a tensor field T of type $(1, 2)$ such that*

$$(AS) \quad \begin{cases} (i) & g(T_X Y, Z) + g(Y, T_X Y) = 0, \\ (ii) & (\nabla_X R)(Y, Z) = [T_X, R_{YZ}] - R_{T_X Y Z} - R_{Y T_X Z}, \\ (iii) & (\nabla_X T)(Y) = [T_X, T_Y] - T_{T_X Y}, \end{cases}$$

and

$$T_X Y + T_Y X = 0, \quad X, Y, Z \in \mathfrak{X}(M),$$

where ∇ denotes the Levi-Civita connection and R is the Riemannian curvature tensor.

II.2. Nonnatural reductivity of the $(H(p, r), g)$ groups.

Theorem 2.2. *The homogeneous manifold $(H(p, r), g)$ is naturally reductive if and only if $H(p, r)$ is a Heisenberg group (i.e., $r = 1$).*

Proof. Suppose that $(H(p, r), g)$ is a naturally reductive homogeneous space. Then there exists a tensor field T of type $(1, 2)$ satisfying the conditions (AS) and such that

$$T_X Y + T_Y X = 0.$$

Let ρ denote the Ricci tensor of the manifold $(H(p, r), g)$. By contraction, from (AS)(ii) we have

$$(\nabla_X \rho)(Y, Z) = -\rho(T_X Y, Z) - \rho(Y, T_X Z).$$

Since T_X is a skew-symmetric operator, the previous condition gives

$$(2.1) \quad \mathfrak{S}_{X, Y, Z}(\nabla_X \rho)(Y, Z) = 0.$$

On the other hand, from (1.3) one has

$$\begin{aligned} (\nabla_{E_i} \rho)(E_\alpha, E_{(\alpha, i)}) &= -(p+r)/4, \\ (\nabla_{E_\alpha} \rho)(E_i, E_{(\alpha, i)}) &= (p+1)/4, \\ (\nabla_{E_{(\alpha, i)}} \rho)(E_\alpha, E_i) &= (1-r)/4. \end{aligned}$$

Now we combine these relations with condition (2.1) and obtain $r = 1$. Conversely, if $r = 1$, the group $H(p, 1)$ is isomorphic to the Heisenberg group H_{2p+1} . In this case we know that (H_{2p+1}, g) is a naturally reductive space for every left-invariant metric g [GP].

III. GEODESIC SYMMETRIES

Let (M, g) be a smooth n -dimensional Riemannian manifold. For every point m in M , consider a neighborhood U_m of m such that for every point $p \in U_m$ there exists a unit vector $\xi \in T_m M$ and a real number r such that $p = \exp_m(r\xi)$. Then the *local geodesic symmetry centered at m* is the diffeomorphism $s_m: U_m \rightarrow U_m$ defined by $s_m(\exp(r\xi)) = \exp(-r\xi)$.

Definition 3.1 [DN]. A Riemannian manifold is a *D'Atri space* if every local geodesic symmetry is volume-preserving (up to sign).

Locally symmetric manifolds are the simplest D'Atri spaces, but there are also a lot of nonsymmetric examples. In particular, all naturally reductive homogeneous Riemannian manifolds are D'Atri spaces. The converse is false. In

[K] Kaplan gives a family of examples of nonnaturally reductive D'Atri spaces. These spaces are connected and simply connected nilpotent Lie groups whose Lie algebras \mathfrak{n} split as $\mathfrak{n} = V \oplus Z$, where Z is the center of \mathfrak{n} , and satisfy

$$g(X, Y) = 0 \quad \text{for all } X \in V, Y \in Z; \quad \|\text{ad}_X^*(Y)\| = \|X_V\| + \|Y_Z\|,$$

where $*$ denotes the adjoint relative to g and X_V (resp. Y_Z) is the projection of X on V (resp. Z). These spaces are called of type H .

It is well known that if an analytic Riemannian manifold is a D'Atri space then it satisfies the Ledger conditions of odd order. Now, the first Ledger condition is equivalent to (2.1). The Riemannian manifold $(H(p, r), g)$ is not a D'Atri space but let D be the distribution given by

$$D = \{(\lambda, \mu, \nu) \in \mathfrak{h}(p, r) \mid \mu_{(\alpha, j)} \nu_j \nu_i + \|\nu\|^2 \mu_{(\alpha, i)} = 0\};$$

then

Theorem 3.2. *Let X be a vector in $\mathfrak{h}(p, r) \setminus D$. Then the local geodesic symmetry with respect to the geodesic through e and determined by X is volume-preserving (up to sign).*

Proof. It follows the same lines as Kaplan's proof.

It is sufficient to consider the geodesic symmetry s at the identity e of $H(p, r)$ along the geodesic γ such that $\dot{\gamma} \in \mathfrak{h}(p, r) \setminus D$. Let $\exp_H: \mathfrak{h}(p, r) \rightarrow H(p, r)$ denote the Lie exponential maps. Let U be a normal neighborhood of e . For X in $\exp^{-1}U$, let $F(X) \in \mathfrak{h}(p, r) \setminus D$ denote the tangent vector at e of the geodesic joining e to $\exp_H X$, and put $\Sigma = F^{-1} \circ (-F)$. Then the geodesic symmetry s maps $\exp_H X$ to $\exp_H \Sigma X$.

The geodesic symmetry s can also be computed easily. Indeed, if we put $F(X) = (a_1, a_2)$ from (1.6) we have

$$X = \frac{(I - e^{J(a_2)})}{\|a_2\|} J(a_2) a_1, \quad Z = \left(1 + \frac{1}{2} \frac{\|a_1\|}{\|a_2\|^2}\right) \left(1 - \frac{\sin\|a_2\|}{\|a_2\|}\right) a_2,$$

and therefore

$$\sum(X) = -e^{-J(a_2)} X, \quad \sum(Z) = -Z.$$

By using conditions (1.4) one gets

$$\begin{aligned} \|X\|^2 &= 2 \frac{\|a_1\|^2}{\|a_2\|^2} (1 - \cos\|a_2\|), \\ \|Z\| &= \left(1 + \frac{1}{2} \frac{\|a_1\|^2}{\|a_2\|^2} \left(1 - \frac{\sin\|a_2\|}{\|a_2\|}\right)\right) \|a_2\|. \end{aligned}$$

For small $\|X\|$, $\|Z\|$, these equations determine $\|a_1\|$, $\|a_2\|$ uniquely, so we can write

$$s(\exp_H X + Z) = \exp_H \left(\sum(X + Z) \right) = \exp_H (-e^{\beta(\|X\|, \|Z\|)J(\|Z\|)} X - Z)$$

for some function β depending only on the length of X and Z .

Although s is not an isometry, it acts isometrically on the spheres of $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ and \mathcal{Z} centered at 0. Therefore, it preserves the Euclidean Lebesgue measure in $\mathfrak{h}(p, r)$. The Riemannian volume element defines a Haar measure

on $H(p, r)$. The Haar measure of a nilpotent group is the exponential of a Lebesgue measure. Finally the geodesic symmetry s is volume-preserving.

IV. LIE ALGEBRAS OF TYPE H AND LIE ALGEBRAS $\mathfrak{h}(p, r)$

The groups of type H and the $H(p, r)$ groups have some common Riemannian properties. Let us prove that they are not the same spaces.

Let H be a group of type H . Its Lie algebra \mathfrak{h} is called an *algebra of type H* . In $[G_2]$ Goze proved the following result:

Proposition 4.1. *Let G be a connected, simply connected n -dimensional two-step nilpotent group with r -dimensional center Z . Then there exists on G a Pfaffian system S of rank r and class n .*

Let $\mathfrak{h}(r)$ be an algebra of type H , where r is the dimension of its center. Then

(1) The Pfaffian system S defines on $\mathfrak{h}(r)$ an *r -contact structure* (an early generalization of contact structure introduced in $[L]$), i.e.,

$$(d\alpha)^{(n-r)/2} = 0 \mod S \quad \forall \alpha \in S.$$

The Pfaffian system S defines on $\mathfrak{h}(p, r)$ an *r -contact system*.

(2) $\mathfrak{h}(r)$ is a model for a Lie algebra with an *r -contact structure*. $\mathfrak{h}(p, r)$ is a model for a Lie algebra with an *r -contact system*.

(3) Every Lie algebra $\mathfrak{h}(r)$, with $\dim \mathfrak{h}(r) = rp + r + p$, can be contracted onto $\mathfrak{h}(p, r)$.

(4) The Engel invariant $c(S)$ for a Pfaffian system (S) satisfies the Gardner inequalities

$$\frac{n-r}{r-1} \leq c(S) \leq \frac{n-r}{2}.$$

For *r -contact systems* we have $c(S) = (n-r)L/(r-1)$; but an *r -contact structure* satisfies $c(S) = (n-r)/2$.

From this follows the

Proposition $[G_2]$. *The Lie algebra $\mathfrak{h}(r)$ is an $\mathfrak{h}(p, r)$ algebra if and only if it is a Heisenberg algebra.*

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