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BOUNDARY BEHAVIOR OF HOLOMORPHIC FUNCTIONS OF $A^p_{q,s}$

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ABSTRACT. In this paper we prove that the Sobolov spaces $A^p_{q,s}(D)$ on bounded strongly pseudoconvex domains D are continuously contained in $\mathrm{BMOA}(\partial D)$ for $0 < p < \infty$, $q \geq 0$, and $s = (n+q)/p$.

1. INTRODUCTION

Let D be a bounded strongly pseudoconvex domain in C^n with smooth boundary ∂D. Let $\delta(z)$ be the distance from z to ∂D, and let $dV_q = C_q\delta(z)^{q-1}dV$ for each $q > 0$, where dV is the $C^n = R^{2n}$ volume element and C_q is chosen so that dV_q is a probability measure on D. As $q \to 0^+$, these measures (as measures on $\overline{D}$) converge to the normalized surface measure on ∂D,which is denoted by dV_0. We use L^p_q for $L^p(dV_q)$ and $\|\cdot\|_{p,q}$ for the L^p_q norm. The space of all holomorphic functions on D satisfying

$$\|f\|_{p,q,s} = \left(\sum_{|\alpha|\leq s}\left\|\frac{\partial^\alpha f}{\partial z^\alpha}\right\|^p_{p,q}\right)^{1/p} < +\infty$$

is denoted by $A^p_{q,s}$ for $0 < p < \infty$, $q \geq 0$, and s a nonnegative integer. For noninteger values of $s > 0$, $A^p_{q,s}$ and $\|\cdot\|_{p,q,s}$ can be defined by interpolation; see [1] for the details. Let BMOA be the space of all holomorphic functions in D whose boundary values are in $\mathrm{BMO}(\partial D)$ (see [9, pp. 235, 253]). The BMO norm is denoted by $\|\cdot\|_{\mathrm{BMO}}$ (strictly speaking, $\|\cdot\|_{\mathrm{BMO}}$ is a norm on functions modulo constants). Let $\mathscr{B}$ be the space of all Bloch functions defined in [6] and $\mathscr{B}_0$ be the little Bloch space of all holomorphic functions satisfying

$$\sup_{\substack{\xi\in T_z(D)\\|\xi|=1}} |f_*(z)\cdot\xi|/F^D_K(z,\xi) \to 0 \quad \text{as } \delta(z)\to 0, \tag{1}$$

where f_* and F^D_K are as defined in [6]. For $\alpha > 0$ let Λ_α be the Lipschitz space of order α [4].

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In [3] Graham proved that $Rf(z) \in A^p_{0,0}(B)$ implies $f \in A^{np/(n-p)}_{0,0}(B)$ if $0 < p < n$ and $f \in \Lambda_{1-n/p}$ if $p > n$, where $B = \{z \in C^n : |z| < 1\}$ and $Rf(z) = \sum_{j=1}^n z_j \partial f/\partial z_j$. Employing analysis on the Heisenberg group, Krantz also obtained these results, and, furthermore, he proved that $Rf(z) \in A^n_{0,0}(B)$ implies $f \in \mathrm{BMOA}$; see [5]. Later on, Beatrous and Burbea proved that, for $0 < p < \infty$ and $q \geq 0$, $A^p_{q,s}(B) \subset A^{p(n+q)/((n+q)-ps)}_{q,0}(B)$ if $0 \leq s < (n+q)/p$, $A^p_{q,s} \subset \mathrm{BOMA} \cap \mathscr{B}_0$ if $s = (n+q)/p$, and $A^p_{q,s} \subset \Lambda_{s-(n+q)/p}$ if $s > (n+q)/p$ (see [2, Theorem 2.7]). In [1] Beatrous proved a theorem which implies that on strongly pseudoconvex domains the inclusion $A^p_{q,s} \subset A^{p(n+q)/((n+q)-ps)}_{q,0}$ is continuous if $s < (n+q)/p$. In this paper we prove

Theorem 1. *Let D be a bounded strongly pseudoconvex domain with smooth boundary, and let $0 < p < \infty$, $q \geq 0$. If $s = (n+q)/p$ then $A^p_{q,s} \subset \mathrm{BMOA} \cap \mathscr{B}_0$ and the inclusion is continuous with respect to the* BMO *norm in* $\mathrm{BMO} \cap \mathscr{B}_0$.

2. Proof of the theorem

If $f \in A^p_{q,s}$, we take three numbers $p_2 > \max(n, p)$, $q_2 \geq 0$, and $s_2 \geq 0$ such that

$$(n + q_2)/p_2 - (n + q)/p = s_2 - s. \tag{2}$$

Using Theorem 1.5(iii) of [1] for $M = D$ we have $f \in A^{p_2}_{q_2,s_2}$ and $\|f\|_{p_2,q_2,s_2} \leq C\|f\|_{p,q,s}$. Observe that $s_2 = (n + q_2)/p_2$ since $s = (n + q)/p$. Applying Theorem 1.2 in [1] we obtain $A^{p_2}_{q_2,s_2} = A^{p_2}_{p_2-n,1}$, and the norms are equivalent. Therefore, it is sufficient to prove that $f \in A^p_{p-n,1}$ $(p > n)$ implies $f \in \mathrm{BMOA} \cap \mathscr{B}_0$ and $\|f\|_{\mathrm{BMO}} \leq C\|f\|_{p,p-n,1}$.

For $z \in D$ near ∂D, let $\Delta(z)$ be a polydisc centered at z with radius $c_1\delta(z)$ in the complex normal direction and radius $c_1\delta(z)^{1/2}$ in $n-1$ orthogonal complex tangential directions (c_1 is fixed and small enough). Since D has smooth boundary, there exists a constant c_2 such that

$$\delta(z)/c_2 \leq \delta(\xi) \leq c_2\delta(z) \quad \text{for } \xi \in \Delta(z).$$

From the plurisubharmonicity of $|\nabla f|^p$ we have

$$\begin{aligned} |\nabla f|^p(z)\delta(z)^p &\leq C\delta(z)^{p-(n+1)} \int_{\Delta(z)} |\nabla f|^p(\xi)\, dV(\xi) \\ &\leq C \int_{\Delta(z)} |\nabla f|^p(\xi)\delta(\xi)^{p-(n+1)}\, dV(\xi) \\ &\leq C \int_{\delta(\xi) \leq c_2\delta(z)} |\nabla f|^p(\xi)\, dV_{p-n}(\xi) \to 0 \quad \text{as } \delta(z) \to 0. \end{aligned}$$

This means that for any $\varepsilon > 0$ there exists $\delta_0 > 0$ such that

$$|\nabla f|(z)\delta(z) < \varepsilon \quad \text{for } \delta(z) < \delta_0. \tag{3}$$

By the argument of Lemma 4.8 in [8], we can prove that, for any complex tangential direction μ at z, the complex tangential derivative $\nabla_\mu f$ satisfies

$$|\nabla_\mu f|(z)\delta(z)^{1/2} \le C\varepsilon \quad \text{for } \delta(z) \le \delta_0/4. \tag{4}$$

By a proof similar to that on p. 152 of [6] we know that (3) and (4) imply (1). Hence, $f \in \mathscr{B}_0$.

Now we are going to prove $f \in \mathrm{BMOA}$. For $f \in A^p_{p-n,1}$, using Corollary 2.3 in [1] we have

$$f(z) = \int_D [f(\zeta)K_0(z, \zeta) + \mathscr{D} f(\zeta)K_1(z, \zeta)]\, dV_{p-n+1}(\zeta),$$

where $\mathscr{D} f = \langle df, \partial\rho\rangle$ (see [1, p. 93]) and K_0 and K_1 are kernels of type p. Let $D_\varepsilon = \{z \in D : \rho(z) \le -\varepsilon\}$ and $d\sigma_\varepsilon$ be the Hausdorff measure of $(2n-1)$ dimensions on ∂D_ε, where $\rho(z)$ is the defining function of D and $\varepsilon > 0$. From Theorem 2.4 in [1] we obtain

$$\begin{aligned}
&\int_{\partial D\varepsilon} |f(z)|\, d\sigma_\varepsilon(z) \\
&\quad \le \int_D (|f(\zeta)| + |\mathscr{D} f(\zeta)|)\, dV_{p-n+1}(\zeta) \int_{\partial D\varepsilon} (|K_0(z, \zeta)| + |K_1(z, \zeta)|)\, d\sigma_\varepsilon(z) \\
&\quad \le C \int_D (|f(\zeta)| + |\mathscr{D} f(\zeta)|)\, \delta(\zeta)^{n-p}\, dV_{p-n+1}(\zeta) \\
&\quad = C \int_D (|f(\zeta)| + |\mathscr{D} f(\zeta)|)\, dV(\zeta) \\
&\quad \le C\left\{ \int_D (|f|^p(\zeta) + |\mathscr{D} f(\zeta)|^p)\delta(\zeta)^{p-n-1}\, dV(\zeta)\right\}^{1/p} \\
&\qquad \times \left\{ \int_D \delta(\zeta)^{p'(n/p-1/p')}\, dV(\zeta)\right\}^{1/p'} \\
&\quad \le C\|f\|_{p,p-n,1} \left\{ \int_D \delta(\zeta)^{p'n/p-1}\, dV(\zeta)\right\}^{1/p'} \le C\|f\|_{p,p-n,1},
\end{aligned}$$

where $1/p + 1/p' = 1$. Hence, f is in the Hardy space $A^1_{0,0} = H^1$. Then we know from [7] that

$$\int_{\partial D} |f(\zeta + \lambda\nu(\zeta)) - f(\zeta)|\, dV_0(\zeta) \to 0 \quad \text{as } \lambda \to 0^+,$$

where $\nu(\zeta)$ is the inward unit normal of ∂D and $f(\zeta)$ is the admissible limit of f at $\zeta \in \partial D$. Therefore, $f(\zeta)$ must be of analytic type (for the definition see [9]). Additionally, we claim that $(|f(z)| + |\nabla f(z)|)\, dV(z)$ is a Carleson measure on D. In fact, for any Carleson window $\widetilde{B}_t(\zeta_0)$ as in [9],

$$\widetilde{B}_t(\zeta_0) = \{B_t(\zeta_0) + \lambda\nu(\zeta_0) : \lambda \in (0, t)\},$$

where $B_t(\zeta_0)$ is the nonisotropic ball of ∂D at ζ_0 with radius t, we have

$$\begin{aligned}
&\int_{\widetilde{B}_t(\zeta_0)} (|f(z)| + |\nabla f(z)|)\, dV(z) \\
&\quad \le \left\{ \int_{\widetilde{B}_t(\zeta_0)} (|f(z)| + |\nabla f(z)|)^p \delta(z)^{p-(n+1)}\, dV(z) \right\}^{1/p} \\
&\qquad \times \left\{ \int_{\widetilde{B}_t(\zeta_0)} \delta(z)^{p((n+1)/p-1)/(p-1)}\, dV(z) \right\}^{(p-1)/p} \\
&\quad \le C\|f\|_{p,p-n,1} \left\{ \int_{\widetilde{B}_t(\zeta_0)} \delta(z)^{(n+1-p)/(p-1)}\, dV(z) \right\}^{(p-1)/p} \\
&\quad \le C\|f\|_{p,p-n,1} \left\{ t^n \int_0^t t^{(n+1-p)/(p-1)}\, dt \right\}^{(p-1)/p} \\
&\quad = C\|f\|_{p,p-n,1} (t^n \cdot t^{n/(p-1)})^{(p-1)/p} = C\|f\|_{p,p-n,1} t^n.
\end{aligned} \tag{5}$$

Therefore, we know from Theorem 2.1.3(ii) of [9] that $f \in \mathrm{BMOA}$. From the proof on p. 253 of [9] and (5), we obtain

$$\|f\|_{\mathrm{BMO}} \le C\|f\|_{p,p-n,1}. \tag{6}$$

This completes the proof.

Remark 1. As mentioned by Varopoulos in [9], Theorem 2.1.3(ii) of [9] is still valid for strongly pseudoconvex domains, but for simplicity's sake it was stated there only in the case $D = \{z \in C^n : |z| < 1\}$.

Remark 2. We can also prove

$$A^p_{q,s} \subset \Lambda_{s-(n+q)/p} \quad \text{for } s > (n+q)/p.$$

In order to prove this assertion, we take $k > s$ a fixed integer. From [1] we have $A^p_{q,s} = A^p_{q+p(k-s),k}$. For $f \in A^p_{q+p(k-s),k}$ and $|\alpha| \le k$

$$\begin{aligned}
\left|\frac{\partial^\alpha f}{\partial z^\alpha}(z)\right|^p &\le C\delta(z)^{-(n+1)-(q+p(k-s)-1)} \int_{\Delta(z)} \left|\frac{\partial^\alpha f}{\partial z^\alpha}(\zeta)\right|^p \delta(\zeta)^{q+p(k-s)-1}\, dV(\zeta) \\
&\le C\|f\|^p_{p,q+p(k-s),k} \cdot \delta(z)^{p((s-(n+q)/p)-k)}.
\end{aligned}$$

Hence,

$$\max_{|\alpha|\le k} \left|\frac{\partial^\alpha f}{\partial z^\alpha}(z)\right| \le C\|f\|_{p,q+p(k-s),k} \cdot \delta(z)^{(s-(n+q)/p)-k}. \tag{7}$$

Applying Theorem 8.8.6 of [4], we have $f \in \Lambda_{s-(n+q)/p}$.

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