ON DIFFEOMORPHISMS OVER T^2 -KNOTS

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ABSTRACT. For a spun T^2 -knot $(S^4, S(k))$ or a twisted spun T^2 -knot $(S^4, \tilde{S}(k))$ of a nontrivial knot k in S^3 , there are infinitely many isotopy classes of embeddings of a 2-torus into a 4-sphere which have this T^2 -knot as their image. This is shown by solving the following question: Which isotopy classes of diffeomorphisms of $S(k)$ or $(S^4, \tilde{S}(k))$ have orientation-preserving diffeomorphisms of S^4 as their extension?

1. INTRODUCTION

In this paper, we work in the differential category.

For an embedding $\varphi : T^2 \hookrightarrow S^4$ whose image is denoted by K , we call (S^4, K) a T^2 -knot. A T^2 -knot (S^4, K) is called *trivial* if K bounds a solid torus $S^1 \times D^2$ in S^4 . Here, we are concerned with the following question:

For any T^2 -knot (S^4, K) how many isotopy classes of embeddings of T^2 into S^4 have this T^2 -knot as their image?

The same kind of question for knots in S^3 is a problem about invertibility of knots. The existence of noninvertible knots was shown by Trotter [T], and this problem was solved for knots with fewer than ten crossings by Hartley [H].

Let $\mathcal{M}(K)$ denote the group of isotopy classes of self-diffeomorphisms on K . An element σ in $\mathcal{M}(K)$ is called *orientation-preserving extensible* (resp. *extensible*) if there exists an orientation-preserving self-diffeomorphism of S^4 (resp. a self-diffeomorphism of S^4) F with $F|_K = \sigma$. Denote the subgroup of $\mathcal{M}(K)$ whose elements are orientation-preserving extensible (resp. extensible) by $\mathcal{E}^+(S^4, K)$ (resp. $\mathcal{E}(S^4, K)$). The number of the isotopy classes which have (S^4, K) as their image is shown to be more than the number of elements of $\mathcal{M}(K)/\mathcal{E}^+(S^4, K)$ by the isotopy extension theorem. Thus it is necessary to determine $\mathcal{E}^+(S^4, K)$.

Let (S^3, k) be any knot (i.e., k is an image of S^1 embedded in S^3) and B be a 3-ball contained in $S^3 \setminus k$. Define $(S^4, S(k))$, $(S^4, \tilde{S}(k))$ (see [I2]) by

$$(S^4, S(k)) = ((S^3, k) \setminus \text{Int } B) \times S^1 \cup_{\text{id}} S^2 \times D^2,$$

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$$(S^4, \tilde{S}(k)) = ((S^3, k) \setminus \text{Int } B) \times S^1 \cup_{\tau} S^2 \times D^2$$

where id is the natural identification of $\partial B \times S^1$ with $S^2 \times \partial D^2$ and τ is the map $(u, v) \mapsto (uv, v)$ where we identify S^2 with the Riemann sphere and D^2 with the unit disk in $\mathbb{C}$. We call $(S^4, S(k))$ the *spun T^2 -knot* of (S^3, k) and call $(S^4, \tilde{S}(k))$ the *twisted spun T^2 -knot* of (S^3, k) . Denote the set of 2×2 matrices over integer by $M(2, \mathbb{Z})$, and denote

$$\text{GL}(2, \mathbb{Z}) = \{A \in M(2, \mathbb{Z}) ; \det A = \pm 1\},$$

$$\text{EGL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{GL}(2, \mathbb{Z}) ; \alpha + \beta + \gamma + \delta \text{ is an even integer} \right\},$$

$$R(\varepsilon_1, \varepsilon_2; 1) = \left\{ \begin{pmatrix} \varepsilon_1 & n \\ 0 & \varepsilon_2 \end{pmatrix} ; n \in \mathbb{Z} \right\},$$

and

$$R(\varepsilon_1, \varepsilon_2; 2) = \left\{ \begin{pmatrix} \varepsilon_1 & 2n \\ 0 & \varepsilon_2 \end{pmatrix} ; n \in \mathbb{Z} \right\}.$$

For a spun T^2 -knot $(S^4, S(k))$ (resp. a twisted spun T^2 -knot $(S^4, \tilde{S}(k))$), let $\{\alpha, \beta\}$ be a basis of $H_1(S(k))$ (resp. $H_1(\tilde{S}(k))$) which is represented by $k \times *$, $* \times S^1$ in $(S^4, S(k))$ (resp. $\tilde{S}(k)$). For $\sigma \in \mathcal{M}(S(k))$ or $\mathcal{M}(\tilde{S}(k))$, if $\sigma_*(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}) = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix})$ then we identify σ with $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \text{GL}(2, \mathbb{Z})$. By this manner, we can identify $\mathcal{M}(S(k))$ and $\mathcal{M}(\tilde{S}(k))$ with $\text{GL}(2, \mathbb{Z})$.

If (S^3, k) is a trivial knot, then $(S^4, S(k))$ is a trivial T^2 -knot. For this trivial T^2 -knot $(S^4, S(k))$, the result that $\mathcal{E}(S^4, S(k))$ is equal to $\text{EGL}(2, \mathbb{Z})$ was shown by Montesinos [M, Theorem 5.4]. By Iwase [I3, Proposition 3.1], for a classical torus knot (S^3, k) , it is shown that $\mathcal{E}^+(S^4, S(k))$ is equal to $R(\pm 1, 1; 2)$ and that $\mathcal{E}^+(S^4, \tilde{S}(k))$ is equal to $R(\pm 1, 1; 1)$. In this paper, we will determine $\mathcal{E}(S^4, S(k))$, $\mathcal{E}^+(S^4, S(k))$, $\mathcal{E}(S^4, \tilde{S}(k))$, and $\mathcal{E}^+(S^4, \tilde{S}(k))$ when (S^3, k) is a nontrivial knot.

Theorem A. *Let (S^3, k) be any nontrivial knot.*

- (a) (1) *If (S^3, k) is an invertible knot or a $(-)$ amphicheiral knot, then $\mathcal{E}(S^4, S(k)) = R(\pm 1, \pm 1; 2)$.*
- (2) *If (S^3, k) is a noninvertible non- $(-)$ amphicheiral knot, then $\mathcal{E}(S^4, S(k)) = R(1, \pm 1; 2)$.*
- (b) (1) *If (S^3, k) is an invertible amphicheiral knot, then $\mathcal{E}^+(S^4, S(k)) = R(\pm 1, \pm 1; 2)$.*
- (2) *If (S^3, k) is an invertible nonamphicheiral knot, then $\mathcal{E}^+(S^4, S(k)) = R(\pm 1, 1; 2)$.*
- (3) *If (S^3, k) is a noninvertible $(+)$ amphicheiral knot, then $\mathcal{E}^+(S^4, S(k)) = R(1, \pm 1; 2)$.*
- (4) *If (S^3, k) is a noninvertible $(-)$ amphicheiral knot, then $\mathcal{E}^+(S^4, S(k)) = R(1, 1; 2) \cup R(-1, -1; 2)$.*
- (5) *If (S^3, k) is a noninvertible nonamphicheiral knot, then $\mathcal{E}^+(S^4, S(k)) = R(1, 1; 2)$.*

Theorem B. *Let (S^3, k) be any nontrivial knot.*

- (a) (1) *If (S^3, k) is an invertible knot or a $(-)$ amphicheiral knot, then $\mathcal{E}(S^4, \tilde{S}(k)) = R(\pm 1, \pm 1; 1)$.*
 (2) *If (S^3, k) is a noninvertible non- $(-)$ amphicheiral knot, then $\mathcal{E}(S^4, \tilde{S}(k)) = R(1, \pm 1; 1)$.*
- (b) (1) *If (S^3, k) is an invertible amphicheiral knot, then $\mathcal{E}^+(S^4, \tilde{S}(k)) = R(\pm 1, \pm 1; 1)$.*
 (2) *If (S^3, k) is an invertible nonamphicheiral knot, then $\mathcal{E}^+(S^4, \tilde{S}(k)) = R(\pm 1, 1; 1)$.*
 (3) *If (S^3, k) is a noninvertible $(+)$ amphicheiral knot, then $\mathcal{E}^+(S^4, \tilde{S}(k)) = R(1, \pm 1; 1)$.*
 (4) *If (S^3, k) is a noninvertible $(-)$ amphicheiral knot, then $\mathcal{E}^+(S^4, \tilde{S}(k)) = R(1, 1; 1) \cup R(-1, -1; 1)$.*
 (5) *If (S^3, k) is a noninvertible nonamphicheiral knot, then $\mathcal{E}^+(S^4, \tilde{S}(k)) = R(1, 1; 1)$.*

By the above theorems, we see:

Corollary. *If k is a non-trivial knot in S^3 , there are infinitely many isotopy classes of embeddings which have $(S^4, S(k))$ as their image, and the same is true for $(S^4, \tilde{S}(k))$.*

2. PRELIMINARIES

A. The Roklin quadratic form. In this section, we quote results from [M].

For any T^2 -knot (S^4, K) , define a quadratic form (*the Roklin quadratic form*) $q_K : H_1(K; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2$ in the following way: Let P be a compact surface embedded in S^4 , with its boundary contained in K , normal to K along its boundary, and its interior transversal to K . Let P' be a surface transverse to P obtained by sliding P parallel to itself over K . Define $q_K([\partial P]) = \#(\text{int } P \cap (P' \cup K)) \bmod 2$. This is a well-defined quadratic form with respect to the homology intersection form on K .

If $\sigma \in \mathcal{E}^+(S^4, K)$ (resp. $\mathcal{E}(S^4, K)$), then $q_K(\sigma_*(x)) = q_K(x)$ for all $x \in H_1(K; \mathbb{Z}_2)$. Thus $\mathcal{E}^+(S^4, K)$ (resp. $\mathcal{E}(S^4, K)$) is included in the subset $H_K = \{\sigma \in \mathcal{M}(K); q_K(\sigma_*(x)) = q_K(x) \text{ for all } x \in H_1(K; \mathbb{Z}_2)\}$. Let $\{\alpha, \beta\}$ be a basis of $H_1(K; \mathbb{Z})$ with $q_K(\alpha) = q_K(\beta) = 0$. Identify $\mathcal{M}(K)$ with $\text{GL}(2, \mathbb{Z})$ by the same way as in §1. With this identification the subgroup H_K is equal to $\text{EGL}(2, \mathbb{Z})$ for any T^2 -knot (S^4, K) (see [M, Proposition 4.3]). A system of generators of $\text{EGL}(2, \mathbb{Z})$ is

$$\left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

For any spun T^2 -knot $(S^4, S(k))$, let $\{\alpha, \beta\}$ be a basis of $H_1(S(k); \mathbb{Z})$ as in §1. It is easy to see that $q_{S(k)}(\alpha) = q_{S(k)}(\beta) = 0$. Hence:

Lemma M. $\mathcal{E}(S^4, S(k))$ and $\mathcal{E}^+(S^4, S(k))$ are included in $\text{EGL}(2, \mathbb{Z})$. $\square$

B. Twins and $(S^4, \tilde{S}(k))$. Let B be a 3-ball contained in $\text{Int } D^2 \times S^1$. Define the *twin* Tw by

$$Tw = (D^2 \times S^1 \setminus \text{Int } B) \times S^1 \cup_{\text{id}} S^2 \times D^2$$

where id is the natural identification of $\partial B \times S^1$ with $S^2 \times \partial D^2$ (see [M, I1]). Let l, r, s be elements in $H_1(\partial(Tw))$ represented by circles $* \times S^1 \times *$, $\partial D^2 \times * \times *$, and $*** \times S^1$ respectively. Then $\{l, r, s\}$ is a basis of $H_1(\partial(Tw))$ (see Figure 1).

Theorem (Montesinos [M, Theorem 5.3]). Let $g : \partial(Tw) \rightarrow \partial(Tw)$ be a diffeomorphism which induces an automorphism on $H_1(\partial(Tw))$,

$$g_* \begin{pmatrix} l \\ r \\ s \end{pmatrix} = \begin{pmatrix} m & n & p \\ a & \alpha & \beta \\ b & \gamma & \delta \end{pmatrix} \begin{pmatrix} l \\ r \\ s \end{pmatrix}.$$

There is a diffeomorphism $G : Tw \rightarrow Tw$ such that $G|_{\partial(Tw)} = g$ if and only if $a = b = 0$ and $\alpha + \beta + \gamma + \delta$ is even. $\square$

Let (S^3, k) be any knot. Let $S^1 \times D^2$ be a solid torus embedded in S^3 with the following conditions:

- (1) The core of this solid torus $S^1 \times 0$ is a trivial knot in S^3 .
- (2) There is $t \in S^1$ such that $t \times D^2$ intersects with k transversely in one point (see Figure 2).

FIGURE 1

FIGURE 2

For $(S^1 \times D^2) \times S^1$, let l_1, r_1, s_1 be elements in $H_1(\partial((S^1 \times D^2) \times S^1))$ represented by circles $* \times \partial D^2 \times *$, $S^1 \times * \times *$, and $* \times * \times S^1$ respectively. Then $\{l_1, r_1, s_1\}$ is a basis of $H_1(\partial((S^1 \times D^2) \times S^1))$ (see Figure 2). Define a diffeomorphism $\tilde{t} : \partial((S^1 \times D^2) \times S^1) \rightarrow \partial((S^1 \times D^2) \times S^1)$ by $\tilde{t}(e^{i\theta}, e^{i\phi}, e^{i\psi}) = (e^{i(\theta+\psi)}, e^{i\phi}, e^{i\psi})$ and a diffeomorphism $t : \partial((S^1 \times D^2) \times S^1) \rightarrow \partial(Tw)$ by $t = \text{id} \circ \tilde{t}$ where id is the natural identification of $\partial((S^1 \times D^2) \times S^1)$ with $\partial(Tw)$.

Then

$$(S^4, \tilde{S}(k)) = Tw \cup_t ((S^1 \times D^2) \times S^1, k \times S^1).$$

Note that $t_* : H_1(\partial((S^1 \times D^2) \times S^1)) \rightarrow H_1(\partial((S^1 \times D^2) \times S^1))$ satisfies

$$t_* \begin{pmatrix} l_1 \\ r_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} l_1 \\ r_1 \\ s_1 \end{pmatrix}.$$

3. THE PROOFS OF THEOREMS A AND B

Two knots (S^3, k_1) and (S^3, k_2) are called *equivalent* if there is an orientation-preserving diffeomorphism $\Phi : (S^3, k_1) \rightarrow (S^3, k_2)$. Denote by $(S^3, -k)$ the knot which differs from (S^3, k) only in its orientation, and denote by (S^3, k^*) the mirror image of (S^3, k) . A knot (S^3, k) is called *invertible* if (S^3, k) is equivalent to $(S^3, -k)$, and (S^3, k) is called *amphicheiral* if (S^3, k) is equivalent to (S^3, k^*) or $(S^3, -k^*)$. A knot (S^3, k) is called *+ amphicheiral* if (S^3, k) is equivalent to (S^3, k^*) , and (S^3, k) is called *- amphicheiral* if (S^3, k) is equivalent to $(S^3, -k^*)$.

In this section, we will show the following lemma.

Lemma 3.1. *Let (S^3, k) be any nontrivial knot.*

- (a) (1) *If (S^3, k) is an invertible knot or a $(-)$ amphicheiral knot, then $\mathcal{E}(S^4, S(k)) \subset R(\pm 1, \pm 1; 2)$ and $\mathcal{E}(S^4, \tilde{S}(k)) \subset R(\pm 1, \pm 1; 1)$.*
- (2) *If (S^3, k) is a noninvertible non- $(-)$ amphicheiral knot, then $\mathcal{E}(S^4, S(k)) \subset R(1, \pm 1; 2)$ and $\mathcal{E}(S^4, \tilde{S}(k)) \subset R(1, \pm 1; 1)$.*
- (b) (1) *If (S^3, k) is an invertible amphicheiral knot, then $\mathcal{E}^+(S^4, S(k)) \subset R(\pm 1, \pm 1; 2)$ and $\mathcal{E}^+(S^4, \tilde{S}(k)) \subset R(\pm 1, \pm 1; 1)$.*
- (2) *If (S^3, k) is an invertible nonamphicheiral knot, then $\mathcal{E}^+(S^4, S(k)) \subset R(\pm 1, 1; 2)$ and $\mathcal{E}^+(S^4, \tilde{S}(k)) \subset R(\pm 1, 1; 1)$.*
- (3) *If (S^3, k) is a noninvertible $(+)$ amphicheiral knot, then $\mathcal{E}^+(S^4, S(k)) \subset R(1, \pm 1; 2)$ and $\mathcal{E}^+(S^4, \tilde{S}(k)) \subset R(1, \pm 1; 1)$.*
- (4) *If (S^3, k) is a noninvertible $(-)$ amphicheiral knot, then $\mathcal{E}^+(S^4, \tilde{S}(k)) \subset R(1, 1; 2) \cup R(-1, -1; 2)$ and $\mathcal{E}^+(S^4, S(k)) \subset R(1, 1; 1) \cup R(-1, -1; 1)$.*
- (5) *If (S^3, k) is a noninvertible nonamphicheiral knot, then $\mathcal{E}^+(S^4, S(k)) \subset R(1, 1; 2)$ and $\mathcal{E}^+(S^4, \tilde{S}(k)) \subset R(1, 1; 1)$.*

Let ι be a homomorphism from $\pi_1(S^3 \setminus k)$ to $\pi_1(S^4 \setminus S(k))$ (resp. to $\pi_1(S^4 \setminus \tilde{S}(k))$) defined by composition of a natural isomorphism from $\pi_1(S^3 \setminus k)$ to $\pi_1((S^3 \setminus k) \setminus \text{int } B^3)$ and a homomorphism from $\pi_1((S^3 \setminus k) \setminus \text{int } B^3)$

to $\pi_1(S^4 \setminus S(k))$ (resp. to $\pi_1(S^4 \setminus \tilde{S}(k))$) induced by inclusion. Then:

Lemma 3.2. *The homomorphism ι is an isomorphism.*

Proof. Let $p_1: (S^3 \setminus k) \times S^1 \rightarrow S^3 \setminus k$ and $p_2: (S^3 \setminus k) \times S^1 \rightarrow S^1$ be projections to first or second factor. A homomorphism $h: \pi_1((S^3 \setminus k) \times S^1) \rightarrow \pi_1(S^3 \setminus k) + \pi_1(S^1)$, which is defined by $h(x) = (p_{1*}(x), p_{2*}(x))$ for all $x \in \pi_1((S^3 \setminus k) \times S^1)$, is an isomorphism. We can see this lemma by Van-Kampen's theorem. $\square$

Let $N \subset S^3$ be a tubular neighborhood of k . Let l, m be loops in $S^3 \setminus k$, included in N . The loop l is called the *longitude* of k if l is homotopic to k in N and homologous to 0 in $S^3 \setminus k$. The loop m is called the *meridian* of k if m is homotopic to zero in N and has linking number $+1$ with k .

Let $N(S(k)) \subset S^4$ be a tubular neighborhood of $S(k)$. Then $N(S(k))$ is a trivial D^2 -bundle over $S(k)$ and $\partial N(S(k))$ is a trivial S^1 -bundle over $S(k)$. By construction of $S(k)$, $\partial N(S(k))$ can be identified with $\partial N \times S^1$, and $(\text{meridian of } k) \times * \subset \partial N \times S^1$ is a fiber of $\partial N(S(k))$. Let μ, λ, ν be elements of $\pi_1(\partial N(S(k)))$, where μ is represented by $(\text{meridian of } k) \times * \subset \partial N \times S^1$, λ is represented by $(\text{longitude of } k) \times * \subset \partial N \times S^1$, and ν is represented by $* \times S^1 \subset \partial N \times S^1$. Then $\{\mu, \lambda, \nu\}$ is a basis of $\pi_1(\partial N(S(k))) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Let $\sigma \in \mathcal{M}(S(k))$ be extensible, i.e., there exists a diffeomorphism $F: S^4 \rightarrow S^4$ such that $F|_{S(k)} = \sigma$. Then $F|_{\partial N(S(k))}$ is a bundle map. If σ is equal to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by an identification in §1, then for an automorphism $(F|_{\partial N(S(k))})_*: \pi_1(\partial N(S(k))) \rightarrow \pi_1(\partial N(S(k)))$ we have

$$(F|_{\partial N(S(k))})_* \begin{pmatrix} \mu \\ \lambda \\ \nu \end{pmatrix} = \begin{pmatrix} \pm 1 & 0 & 0 \\ s & a & b \\ t & c & d \end{pmatrix} \begin{pmatrix} \mu \\ \lambda \\ \nu \end{pmatrix}$$

with integers s, t . As an element of $H_1(S^4 \setminus S(k))$, $\lambda = \nu = 0$ and $\mu \neq 0$. Hence $s = t = 0$. As an element of $\pi_1(S^4 \setminus S(k))$, $\nu = 0$, $\mu \neq 0$, and $\lambda \neq 0$. Hence $c = 0$. Hence σ is included in $R(\pm 1, \pm 1 : 1)$. In the same manner we can show that $\sigma \in \mathcal{E}(S^4, \tilde{S}(k))$ is included in $R(\pm 1, \pm 1 : 1)$. By Lemma M, $\sigma \in \mathcal{E}(S^4, S(k))$ is included in $\text{EGL}(2, \mathbb{Z})$; therefore, σ is equal to $\begin{pmatrix} \pm 1 & 2n \\ 0 & \pm 1 \end{pmatrix}$ with an integer n and proper signs of ± 1 . The same is not true for $\sigma \in \mathcal{E}(S^4, \tilde{S}(k))$ since, for the basis $\{\alpha, \beta\}$ of $H_1(S^4, \tilde{S}(k))$ in §1, $q_{S(k)}^{\sim}(\alpha) = 0$ but $q_{S(k)}^{\sim}(\beta) = 1$.

If $\begin{pmatrix} \varepsilon_1 & n \\ 0 & \varepsilon_2 \end{pmatrix}$ is included in $\mathcal{E}^+(S^4, S(k))$ or $\mathcal{E}^+(S^4, \tilde{S}(k))$, then there exists an orientation-preserving diffeomorphism $F: S^4 \rightarrow S^4$ such that $F|_{S(k)} = \sigma$ or $F|_{(S^4, \tilde{S}(k))_t} = \sigma$. Then for an automorphism $(F|_{\partial N(S(k))})_*: \pi_1(\partial N(S(k))) \rightarrow \pi_1(\partial N(S(k)))$ we have

$$(F|_{\partial N(S(k))})_* \begin{pmatrix} \mu \\ \lambda \\ \nu \end{pmatrix} = \begin{pmatrix} \varepsilon_1 \varepsilon_2 & 0 & 0 \\ 0 & \varepsilon_1 & n \\ 0 & 0 & \varepsilon_2 \end{pmatrix} \begin{pmatrix} \mu \\ \lambda \\ \nu \end{pmatrix}.$$

Lemma 3.1 follows from the following theorem of Waldhausen [W].

Theorem (Waldhausen). *Let (S^3, k) be any knot. Denote the longitude of k by λ , and the meridian of k by μ .*

(1) (S^3, k) is an invertible knot if and only if there exists an automorphism $\psi : \pi_1(S^3 \setminus k) \rightarrow \pi_1(S^3 \setminus k)$ such that $\psi(\lambda) = \lambda^{-1}$, $\psi(\mu) = \mu^{-1}$.

(2) (S^3, k) is a $+$ amphicheiral (resp. $-$ amphicheiral) knot if and only if there exists an automorphism $\psi : \pi_1(S^3 \setminus k) \rightarrow \pi_1(S^3 \setminus k)$ such that $\psi(\lambda) = \lambda$, $\psi(\mu) = \mu^{-1}$ (resp. $\psi(\lambda) = \lambda^{-1}$, $\psi(\mu) = \mu$).

Proof. This is an immediate result from Corollary 6.5 in [W]. $\square$

We devote the rest of this paper to proving the following lemma and completing the proofs of Theorems A and B.

Lemma 3.3. *Let (S^3, k) be any nontrivial knot.*

- (a) (1) If (S^3, k) is an invertible knot or a $(-)$ amphicheiral knot, then $\mathcal{E}(S^4, S(k)) \supset R(\pm 1, \pm 1; 2)$ and $\mathcal{E}(S^4, \tilde{S}(k)) \supset R(\pm 1, \pm 1; 1)$.
 (2) If (S^3, k) is a noninvertible non- $(-)$ amphicheiral knot, then $\mathcal{E}(S^4, S(k)) \supset R(1, \pm 1; 2)$ and $\mathcal{E}(S^4, \tilde{S}(k)) \supset R(1, \pm 1; 1)$.
- (b) (1) If (S^3, k) is an invertible amphicheiral knot, then $\mathcal{E}^+(S^4, S(k)) \supset R(\pm 1, \pm 1; 2)$ and $\mathcal{E}^+(S^4, \tilde{S}(k)) \supset R(\pm 1, \pm 1; 1)$.
 (2) If (S^3, k) is an invertible nonamphicheiral knot, then $\mathcal{E}^+(S^4, S(k)) \supset R(\pm 1, 1; 2)$ and $\mathcal{E}^+(S^4, \tilde{S}(k)) \supset R(\pm 1, 1; 1)$.
 (3) If (S^3, k) is a noninvertible $(+)$ amphicheiral knot, then $\mathcal{E}^+(S^4, S(k)) \supset R(1, \pm 1; 2)$ and $\mathcal{E}^+(S^4, \tilde{S}(k)) \supset R(1, \pm 1; 1)$.
 (4) If (S^3, k) is a noninvertible $(-)$ amphicheiral knot, then $\mathcal{E}^+(S^4, S(k)) \supset R(1, 1; 2) \cup R(-1, -1; 2)$ and $\mathcal{E}^+(S^4, \tilde{S}(k)) \supset R(1, 1; 1) \cup R(-1, -1; 1)$.
 (5) If (S^3, k) is a noninvertible nonamphicheiral knot, then $\mathcal{E}^+(S^4, S(k)) \supset R(1, 1; 2)$ and $\mathcal{E}^+(S^4, \tilde{S}(k)) \supset R(1, 1; 1)$.

A. Spun T^2 -knots.

Lemma 3.4. *For any knot (S^3, k) , $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is included in $\mathcal{E}(S^4, S(k))$.*

Proof. Let $\tau \in \mathcal{M}(S(k))$ be represented by a Dehn twist about $* \times S^1 \subset S(k)$ (see Figure 3). Then τ^2 is identified to $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Let $\varphi : T^2 \hookrightarrow S^4$ be an

FIGURE 3

embedding with its image $S(k)$. It suffices to see that $\tau^2 \circ \varphi$ is isotopic to φ , because of the isotopy extension theorem. Let $k \subset S^3 \setminus \text{int } B^3$ and the properly embedded 2-disk $D^2 \subset S^3 \setminus \text{int } B^3$ intersect transversely except in an interval $I \subset k$ which is included in D^2 (see Figure 4). We get $S^3 \subset S^4$ by

$$S^3 = D^2 \times S^1 \cup_{\text{id}} \partial D^2 \times D^2 \subset (S^3 \setminus \text{int } B^3) \times S^1 \cup_{\text{id}} S^2 \times D^2 = S^4.$$

For this 3-sphere, it follows that $S^3 \cap S(k)$ consists of S^1 's and $I \times S^1$. In the following manner, we can construct an isotopy between φ and $\tau^2 \circ \varphi$ with its support in $I \times S^1$. First, isotope $I \times S^1$ in S^3 (Figure 5(a) to Figure 5(c)). Let $N(S^3)$ be a tubular neighborhood in S^4 . By construction of S^3 ,

FIGURE 4

FIGURE 5

$N(S^3)$ is identified with $[-1, 1] \times S^3$, where S^3 is identified with $0 \times S^3$. Let $t: N(S^3) = [-1, 1] \times S^3 \rightarrow [-1, 1]$ be a projection to the first factor. Isotope $I \times S^1$ only in the first factor of $[-1, 1] \times S^3$ such that outside of the annulus A of Figure 5(c) $t = 0$, and inside $t > 0$. Isotope $I \times S^1$ only in the second factor of $[-1, 1] \times S^3$ so that, when the annulus A is pushed down to $0 \times S^3$, $I \times S^1$ is as in Figure 5(d). Finally, isotope this to Figure 5(f). Composing these isotopies, we get the isotopy between φ and $\tau^2 \circ \varphi$. $\square$

By the above lemma, we can see that the subgroup $\{(\begin{smallmatrix} 1 & 2n \\ 0 & 1 \end{smallmatrix}); n \in \mathbb{Z}\}$ is included in $\mathcal{E}^+(S^4, S(k))$ and $\mathcal{E}(S^4, S(k))$. Hence we need to see the extensibility of $\{(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}), (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})\}$. But it follows from the easy construction of the diffeomorphisms on $(S^4, S(k))$.

B. Twisted spun T^2 -knots.

Lemma 3.5. *For any knot (S^3, k) , $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ is included in $\mathcal{E}(S^4, \tilde{S}(k))$ and $\mathcal{E}^+(S^4, \tilde{S}(k))$.*

Proof. From condition (2) in §2B for a solid torus $S^1 \times D^2$ in S^3 there is a sufficiently small number $\varepsilon > 0$ such that, for all $\theta \in [-\varepsilon, \varepsilon]$, $te^{i\theta} \times D^2$ intersect k transversely in one point. Let the diffeomorphism $f: (S^1 \times 0) \times S^1 \rightarrow (S^1 \times 0) \times S^1$ be the Dehn twist along $(t \times 0) \times S^1$ with its support in $(te^{i[-\varepsilon, \varepsilon]} \times 0) \times S^1$ (see Figure 6). Define the map $\tilde{f}: S^1 \times S^1 \rightarrow S^1$ by $f(t, 0, s) = (t, 0, \tilde{f}(t, s))$ for $t, s \in S^1$. Define the diffeomorphism $F: (S^1 \times D^2) \times S^1 \rightarrow (S^1 \times D^2) \times S^1$ by $F(t, x, s) = (t, x, \tilde{f}(t, s))$ for $t, s \in S^1, x \in D^2$; then

$$(F|_{\partial((S^1 \times D^2) \times S^1)})_* \begin{pmatrix} l_1 \\ r_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} l_1 \\ r_1 \\ s_1 \end{pmatrix},$$

$F|_{\tilde{S}(k)}$ is equal to $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$. Following the construction of $(S^4, \tilde{S}(k))$ in §2B, we consider $F|_{\partial((S^1 \times D^2) \times S^1)}$ as a diffeomorphism over ∂Tw ; then

$$(F|_{\partial Tw})_* \begin{pmatrix} l \\ r \\ s \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} l \\ r \\ s \end{pmatrix}.$$

FIGURE 6

By Montesinos's theorem in §2B, there is a diffeomorphism $G : Tw \rightarrow Tw$ such that $G|_{\partial Tw} = F|_{\partial Tw}$. The lemma follows. $\square$

Remark. The fact that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is included in $\mathcal{E}(S^4, S(k))$ and $\mathcal{E}^+(S^4, S(k))$ was shown in the same manner as in the above lemma. But note that $(S^4, S(k)) = Tw \cup_{\text{id}} ((S^1 \times D^2) \times S^1, k \times S^1)$.

For $\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \}$, their extensibility follows from the easy construction of diffeomorphism on $(S^4, \tilde{S}(k))$.

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