OPERATORS WITH NILPOTENT p -CURVATURE

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ABSTRACT. This article studies the variety of nilpotent p -curvature of linear differential operators on the sphere with $m + 1$ singularities. We shall show that for a given set of exponents $\in \mathbb{F}_p$ satisfying Fuchs's relation the associated variety of nilpotent p -curvature is a nonempty complete intersection of dimension, essentially, $m - 2$. The subject has previously been studied by Katz, Honda, and Dwork.

1. GENERALITIES

Let p be a prime, $\mathfrak{F}$ a field of characteristic p that is algebraically closed, $\mathfrak{K} = \mathfrak{F}(x)$ the field of rational functions over $\mathfrak{F}$, $D = \frac{d}{dx}$, $L = D^n + p_1(x)D^{n-1} + \cdots + p_n(x)$ a linear operator with coefficients $p_j(x) \in \mathfrak{K}$, and $\mathfrak{A} = \mathfrak{K}[D]$ the noncommutative ring of operators with coefficients in $\mathfrak{K}$.

Definition. The operator L has nilpotent p -curvature if L splits in $\mathfrak{A}$, i.e., $L = (D + q_1) \cdots (D + q_n)$, $q_j \in \mathfrak{K}$, and $(D + q_j)y_j = 0$ has a nontrivial solution $y_j \in \mathfrak{K}$, $1 \leq j \leq n$.

An equivalent definition is: The operator L has nilpotent p -curvature if $D^{pn} \in \mathfrak{A}L$ (see [Dw, Lemma 0.6]).

The indicial polynomial $f_a(s)$ of an operator L at a point a is the coefficient of the term of lowest order in the Laurent expansion of $L(x - a)^s$ at $x = a$. The characteristic polynomial at infinity f_∞ is defined in a similar way. The roots of $f(s) = 0$ are called exponents.

Theorem 1 (Honda). *Let $\mathfrak{F}$ be a field of characteristic p that is algebraically closed, $\mathfrak{K} = \mathfrak{F}(x)$, and $D = \frac{d}{dx}$. Let $L = D^n + p_1(x)D^{n-1} + \cdots + p_n(x)$ be an operator with nilpotent p -curvature with coefficients in $\mathfrak{K}$. Then the singularities of L are Fuchsian, i.e., $(x - a)^j p_j(x)$ has no singularity at a and $p_j(x)$ has a zero at ∞ of order at least j . Moreover, the exponents lie in $\mathbb{F}_p$.*

Proof. If $n = 1$, then, since $L = D + w$ has a nonzero solution $u \in \mathfrak{F}$, $Lu = Du + wu = 0$. Therefore, $w = -Du/u = e_1/(x - a_1) + \cdots + e_m/(x - a_m)$, $e_i \in \mathbb{F}_p$, $a_i \in \mathfrak{F}$. If $n \geq 2$, then L splits in $\mathfrak{A}$,

$$L = (D + q)(D^{n-1} + r_1 D^{n-2} + \cdots + r_{n-1}).$$

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By comparing coefficients we get:

$$\begin{aligned} p_1 &= q + r_1, \\ p_2 &= qr_1 + Dr_1 + r_2, \\ &\vdots \\ p_{n-1} &= qr_{n-2} + Dr_{n-2} + r_{n-1}, \\ p_n &= qr_{n-1} + Dr_{n-1}, \end{aligned}$$

and the first part of the theorem follows by induction. We write $L = L_1 \circ L_2$, L_2 being of order 1. Then $f_L(s) = f_{L_1}(s-1)f_{L_2}(s)$, and the assertion follows again by induction since it is true for $n = 1$.

Let $L = D^n + p_1D^{n-1} + \dots + p_n$ be an operator that has m distinct finite singularities. If all $m+1$ singularities are Fuchsian then L is called a Fuchsian operator and we have the following two well-known lemmata.

Lemma 2 (Fuchs's Relation). *The exponents of a Fuchsian operator L of order n with $m+1$ singularities satisfy $\sum_{i=1}^{m+1} \sum_{j=1}^n s_{ij} \equiv (m-1)\binom{n}{2} \pmod{p}$.*

Lemma 3. *If L is a Fuchsian operator of order n with $m = 1$ finite singularity a_1 then L is necessarily of the form*

$$L = \frac{1}{(x-a_1)^n} \prod_{j=1}^n \left((x-a_1) \frac{d}{dx} - s_{1,j} \right)$$

(Euler's equation).

We assemble in this paragraph some more terminology. Let $n \geq 2$ be the order of a monic operator $L \in \mathfrak{R}$, and let the number of singularities be at most $m+1$, where $m \geq 1$. For $m \geq 2$ let $\mathbf{s} = (s_{ij})$ be an $(m+1) \times n$ matrix with coefficients $s_{ij} \in \mathbb{F}_p$ such that $\sum_{i=1}^{m+1} \sum_{j=1}^n s_{ij} \equiv (m-1)\binom{n}{2} \pmod{p}$. Let $(s)_j = \prod_{l=0}^{j-1} (s+l)$ and $[s]_j = \prod_{l=0}^{j-1} (s-l)$. We define c_{ij} , $i = 1, \dots, m+1$, $j = 1, \dots, n$, as follows:

$$\begin{aligned} \prod_{j=1}^n (s - s_{ij}) &= [s]_n + \sum_{j=1}^n c_{ij} \cdot [s]_{n-j}, \quad i = 1, \dots, m, \\ \prod_{j=1}^n (s - s_{m+1,j}) &= (s)_n + (-1)^j \cdot \sum_{j=1}^n \left(c_{m+1,j} + \sum_{i=1}^m c_{ij} \right) \cdot (s)_{n-j}. \end{aligned}$$

Note that $c_{ij} \in \mathbb{F}_p$. It follows from Fuchs's relation that $c_{m+1,1} = 0$. Let $\mathbf{a} = (a_i)$, $a_i \in \mathfrak{F}$, $i = 1, \dots, m$. We do not require that the a_i be distinct or that the a_i be true singularities. Let $\vartheta = \prod_{i=1}^m (x-a_i)$. Let $k(j) = m(j-1) - j$, $j \geq 1$, let $(r_{j,l})_{1 \leq l \leq k(j)} \in \mathfrak{F}^{k(j)}$, $j \geq 2$, and let $\mathbf{r} = (r_{j,l})_{1 \leq l \leq k(j), 2 \leq j \leq n} \in \mathfrak{F}^N$, where $N = \sum_{j=2}^n k(j) = \sum_{j=2}^n (m(j-1) - j) = (n-1)(n(m-1) - 2)/2$. Then

we define an operator $L = L_{\mathbf{a}, \mathbf{r}, \mathbf{s}}$ as

$$\begin{aligned} L &= D^n + \left(\sum_{i=1}^m \frac{c_{i,1}}{x - a_i} \right) D^{n-1} \\ &\quad + \sum_{j=2}^n \left(\sum_{i=1}^m \frac{c_{ij}}{(x - a_i)^j} + \frac{c_{m+1,j} x^{k(j)} + \sum_{l=0}^{k(j)-1} r_{j,l} x^l}{\vartheta^{j-1}} \right) D^{n-j} \\ &= D^n + \frac{P_1}{\vartheta} D^{n-1} + \cdots + \frac{P_n}{\vartheta^n} = D^n + p_1 D^{n-1} + \cdots + p_n. \end{aligned}$$

Note that $P_j \in \mathbb{F}_p[\mathbf{a}, \mathbf{r}, x]$ and $p_j \in \mathbb{F}_p[\mathbf{a}, \mathbf{r}, x, 1/\vartheta]$. For convenience we shall also use the notation

$$L = D^n + \sum_{j=1}^n \left(\sum_{i=1}^m \frac{c_{ij}}{(x - a_i)^j} + \frac{c_{m+1,j} x^{k(j)} + \sum_{l=0}^{k(j)-1} r_{j,l} x^l}{\vartheta^{j-1}} \right) D^{n-j}$$

with the understanding that for $j = 1$ the second term in parentheses is $= 0$ and the $r_{1,l}$ do not exist ($c_{m+1,1} = 0$ and $k(1) = -1$, so the sum is empty).

$$(\mathbf{a}, \mathbf{s}) = \begin{pmatrix} a_1 & \cdots & a_m & \infty \\ s_{1,1} & \cdots & s_{m,1} & s_{m+1,1} \\ \vdots & & \vdots & \vdots \\ s_{1,n} & \cdots & s_{m,n} & s_{m+1,n} \end{pmatrix}$$

is called a Riemann scheme. If we fix m, n , and $\mathbf{s}$, then the operator $L = L_{\mathbf{a}, \mathbf{r}, \mathbf{s}}$ is parametrized by the N accessory parameters $\mathbf{r} = (r_{j,l})$ and by $\mathbf{a} = (a_i)$. If the a_i are distinct then $L_{\mathbf{a}, \mathbf{r}, \mathbf{s}}$ is a differential operator of Fuchsian type with singularities $\subset \{a_1, \dots, a_m, \infty\}$ and $(\mathbf{a}, \mathbf{s})$ corresponds to the classical Riemann scheme, i.e., the s_{ij} are the exponents of L . If the a_i are not distinct, then L may not be Fuchsian and the s_{ij} may not be exponents. We note that $(\mathbf{a}, \mathbf{r}) \mapsto L_{\mathbf{a}, \mathbf{r}, \mathbf{s}}$ is a mapping of $\mathfrak{F}^m \times \mathfrak{F}^N$ into all Fuchsian operators in $\mathfrak{A}$ of order n with at most m finite singularities and exponent scheme $\mathbf{s}$. In general, an operator does not determine a unique $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ such that $L = L_{\mathbf{a}, \mathbf{r}, \mathbf{s}}$, but $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ is unique if $\prod_{1 \leq i < i' \leq m} (a_i - a_{i'}) \neq 0$. We denote by $S(m, n)$ the set of all $(m+1) \times n$ matrices with coefficients in $\mathbb{F}_p$ that satisfy Fuchs's relation. We denote by $W = W(m, n, \mathbf{s})$ the set of $(\mathbf{a}, \mathbf{r})$ such that $L_{\mathbf{a}, \mathbf{r}, \mathbf{s}}$ has nilpotent p -curvature. We denote by $V = V(m, n, \mathbf{s})$ the set of $(\mathbf{a}, \mathbf{r}) \in W$ such that $\prod_{1 \leq i < i' \leq m} (a_i - a_{i'}) \neq 0$. Since L has nilpotent p -curvature if and only if $D^{pn} \equiv 0 \pmod{\mathfrak{A}L}$, V and W are algebraic sets.

The definition of L for $m = 1$ is motivated by Lemma 3. Let $s_{1,j} \in \mathbb{F}_p$, $j = 1, \dots, n$, and set $s_{2,j} = -s_{1,j}$. The c_{ij} are defined as before. There are no accessory parameters, and if we fix the finite singularity a there is only one operator $L = L_{a, \cdot, \mathbf{s}}$ given by Lemma 3. In this case $V = W = \mathfrak{F}$.

For future reference we state:

Theorem 4. *If $(\mathbf{a}, \mathbf{r}) \in V$ then $\mathbf{r}$ is integral over $\mathbb{F}_p[\mathbf{a}]$.*

Proof. This is a direct consequence of (1.4) in [Dw, p. 759]. In the terminology of that article

$$L = D^n + \left(\sum_{j=1}^n \frac{\sum_{i=1}^m c_{ij} \frac{\vartheta^l(a_i)^{j-1}}{x - a_i} + c_{m+1,j} x^{k(j)} + \sum_{l=0}^{k(j)-1} v_{j,l} x^l}{\vartheta^{j-1}} \right) D^{n-j}$$

where the $v_{j,l}$ are the original accessory parameters of Klein. Therefore,

$$\sum_{l=0}^{k(j)-1} (v_{j,l} - r_{j,l})x^l = \sum_{i=1}^m c_{ij} \left(\frac{\vartheta(x)^{j-1}}{(x-a_i)^j} - \frac{\vartheta'(a_i)^{j-1}}{x-a_i} \right),$$

so $v_{j,l} - r_{j,l} \in \mathbb{F}_p[\mathbf{a}]$, and since the $v_{j,l}$ are shown to be integral over $\mathbb{F}_p[\mathbf{a}]$ in the reference, the result follows.

Corollary. *If we fix $\mathbf{a}$ then V is finite.*

2. THE CONDITIONS FOR THE ACCESSORY PARAMETERS

Let $L = D^n + P_1/\vartheta \cdot D^{n-1} + \cdots + P_n/\vartheta^n = D^n + p_1 D^{n-1} + \cdots + p_n$, and let $\mathfrak{K} = \mathfrak{K}[D]$. Define $b_{l,i}$ by $D^l \equiv b_{l,1} D^{n-1} + b_{l,2} D^{n-2} + \cdots + b_{l,n} \pmod{\mathfrak{K}L}$. Note that $b_{l,i} \in \mathbb{F}_p[\mathbf{a}, \mathbf{r}, x, 1/\vartheta]$. If u is a nonzero solution of L in some extension field and $\mathbf{u} = (u, Du, \dots, D^{n-1}u)$ then

$$\begin{aligned} D^l \mathbf{u} &= D^l(u, \dots, D^{n-1}u) = (D^l u, \dots, D^{l+n-1}u) \\ &= (u, \dots, D^{n-1}u) \begin{pmatrix} b_{l,n} & \cdots & b_{l+n-1,n} \\ \vdots & & \vdots \\ b_{l,1} & \cdots & b_{l+n-1,1} \end{pmatrix} = \mathbf{u} G_l, \end{aligned}$$

where G_l is an $n \times n$ matrix with coefficients in $\mathfrak{K}$. Since $u G_{p(l+1)} = D^{p(l+1)}u = D^p(u G_{pl}) = (D^p u) G_{pl} = u G_p G_{pl}$ by Leibnitz's rule and the annihilation of $\mathfrak{K}$ by D^p , $G_{pl} = G_p^l$. Let $u_1, \dots, u_n$ be solutions of $Lu = 0$ in a differential extension field, independent over the field of constants, and

$$Y = \begin{pmatrix} u_1 & \cdots & D^{n-1}u_1 \\ \vdots & & \vdots \\ u_n & \cdots & D^{n-1}u_n \end{pmatrix}.$$

Then we get a condition for nilpotent p -curvature:

$$\begin{aligned} L \text{ has nilpotent } p\text{-curvature} &\Leftrightarrow D^{pn} \in \mathfrak{K}L \\ &\Leftrightarrow D^{pn} Y = Y G_{pn} = Y G_p^n \neq 0 \\ &\Leftrightarrow G_p \text{ is nilpotent as a matrix} \\ &\Leftrightarrow \text{ch}(G_p)(y) = \det(G_p - yI) = (-y)^n. \end{aligned}$$

Theorem 5. *The characteristic polynomial $\text{ch}(G_p)$ has coefficients in $\mathfrak{F}(x^p)$.*

Proof. By definition, Y is nonsingular and satisfies $DY = YG$. Using that $DY^{-1} = -Y^{-1}(DY)Y^{-1} = -Y^{-1}YGY^{-1} = -GY^{-1}$ we get

$$D(YG_p Y^{-1}) = D(YG_p)Y^{-1} + YG_p DY^{-1} = YG_p GY^{-1} - YG_p GY^{-1} = 0.$$

But $D(YG_p Y^{-1}) = 0$ implies $D(\text{ch}(YG_p Y^{-1})) = 0$, and this is equivalent to $D(\text{ch}(G_p)) = 0$, which is the assertion.

We denote the coefficients of the characteristic polynomial of G_p by C_j :

$$\text{ch}(G_p)(y) = (-y)^n + C_1(-y)^{n-1} + \cdots + C_n, \quad C_j \in \mathfrak{F}(x^p).$$

From the recursion relation

$$G_{l+1} = (D + G)G_l = G'_l + \begin{pmatrix} 0 & \cdots & 0 & 0 & -p_n \\ 1 & \cdots & 0 & 0 & -p_{n-1} \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & -p_2 \\ 0 & \cdots & 0 & 1 & -p_1 \end{pmatrix} \cdot G_l$$

(or $b_{l+1,j} = b'_{l,j} + b_{l,j+1} - b_{l,1} \cdot p_j$, where $b_{l,n+1} = 0$) we obtain by induction that the order of a pole of $b_{l,j}$ is $\leq l + j - n$ or, in matrix notation, $\vartheta^p G_p = \Delta Q \Delta^{-1}$ where Q is a matrix with coefficients $\in \mathbb{F}_p[\vartheta, \vartheta', \dots, \vartheta^{(p-1)}, P_1, P'_1, \dots, P_1^{(p-1)}, \dots, P_n, P'_n, \dots, P_n^{(p-1)}] \subset \mathbb{F}_p[\mathbf{a}, \mathbf{r}, x]$ and

$$\Delta = \text{diag}(1, \vartheta, \dots, \vartheta^{n-1}).$$

For the order of pole of C_j we have

Theorem 6. *If $\mathbf{s} \in S(m, n)$ then $\vartheta^{p(j-1)} C_j \in \mathbb{F}_p[\mathbf{a}, \mathbf{r}, x^p]$ and $\deg_x \vartheta^{p(j-1)} C_j < pm(j-1) - pj = p \cdot k(j)$.*

Proof. Assume first that $a_1, \dots, a_m$ are distinct. Suppose the operator $L = D^n + P_1/\vartheta \cdot D^{n-1} + \dots + P_n/\vartheta^n = D^n + p_1 D^{n-1} + \dots + p_n$ has a singularity $a_1 = 0$. For simplicity we drop unnecessary subscripts and use c_j instead of $c_{1,j}$ and s_j instead of $s_{1,j}$. Then $p_j = c_j/x^j + g_j(x)$, where $g_j(x) \in \mathfrak{K} = \mathfrak{F}(x)$, and $x^{j-1}g_j(x)$ has no singularity at 0. Similarly, we write $b_{l,j} = d_{l,j}/x^{l+j-n} + h_{l,j}(x)$, where $h_{l,j}(x) \in \mathfrak{K}$ and $x^{l+j-n-1}h_{l,j}(x)$ has no singularity at 0. Let $\widehat{C}_j = \vartheta^{pj} C_j$. $\widehat{C}_j(x^p) \in \mathbb{F}_p[\mathbf{a}, \mathbf{r}, x^p]$. We claim that $x \mid \widehat{C}_j$. Let

$$\mathbf{b}_l = \begin{pmatrix} b_{l,n} \\ \vdots \\ b_{l,1} \end{pmatrix}, \quad \mathbf{d}_l = \begin{pmatrix} d_{l,n} \\ \vdots \\ d_{l,1} \end{pmatrix},$$

$$T = \begin{pmatrix} 1 & & & 0 \\ & x & & \\ & & \ddots & \\ 0 & & & x^{n-1} \end{pmatrix}, \quad U = \begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & n-1 \end{pmatrix},$$

$H_l = (\mathbf{d}_l \cdots \mathbf{d}_{l+n-1})$, and $H = H_1$. Then $G_l = (\mathbf{b}_l \cdots \mathbf{b}_{l+n-1})$, $\mathbf{b}_l = Tx^{-l}(\mathbf{d}_l + O(x))$, and $G_l = Tx^{-l}(H_l + O(x))T^{-1}$ where $F(x) = O(x)$ means $F(x) \in x\mathfrak{F}[[x]]$. Then,

$$\begin{aligned} \mathbf{b}_{l+1} &= (D + G)\mathbf{b}_l \\ &= D(Tx^{-l}(\mathbf{d}_l + O(x))) + Tx^{-l}(H + O(x))x^{-l}(\mathbf{d}_l + O(x)) \\ &= Tx^{-l-1}((-lI + U + H)\mathbf{d}_l + O(x)) \\ &= Tx^{-l-1}(\mathbf{d}_{l+1} + O(x)). \end{aligned}$$

Thus, $\mathbf{d}_{l+1} = (-lI + U + H)\mathbf{d}_l$ and

$$\begin{aligned}
 H_p &= (\mathbf{d}_p \cdots \mathbf{d}_{p+n-1}) \\
 &= \left(\prod_{l=0}^{p-1} (-lI + U + H)\mathbf{d}_0 \cdots \prod_{l=n-1}^{p+n-2} (-lI + U + H)\mathbf{d}_{n-1} \right) \\
 &= \prod_{l=0}^{p-1} (-lI + U + H) \cdot (\mathbf{d}_0 \cdots \mathbf{d}_{n-1}) \\
 &= \prod_{l=0}^{p-1} (-lI + U + H) \cdot I = (U + H)^p - (U + H).
 \end{aligned}$$

The characteristic polynomial of $U + H$ is

$$\begin{aligned}
 \text{ch}(U + H) &= \left| \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -c_n \\ 1 & 1 & \cdots & 0 & 0 & -c_{n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & n-3 & 0 & -c_3 \\ 0 & 0 & \cdots & 1 & n-2 & -c_2 \\ 0 & 0 & \cdots & 0 & 1 & n-1-c_1 \end{pmatrix} - yI \right| \\
 &= (n-1-y)(n-2-y) \cdots (-y) - c_1(n-2-y) \cdots (-y) \\
 &\quad + c_2(n-3-y) \cdots (-y) + \cdots + (-1)^n c_n \\
 &= \prod_{j=1}^n (s_j - y).
 \end{aligned}$$

Now $\text{ch}(x^p G_p) = \text{ch}(T(H_p + O(x))T^{-1}) = \text{ch}(H_p) + O(x)$ and, therefore, showing that $x \mid \vartheta^{pj} C_j$ is equivalent to showing that $\text{ch}(H_p) = (-y)^n$.

$$\text{ch}(H_p) = \text{ch}((U + H)^p - (U + H)) = \prod_{j=1}^n (s_j^p - s_j - y) = (-y)^n.$$

(Note that $s_j \in \mathbb{F}_p$.) Since the calculation is the same at each a_i , we now have proved that $\vartheta \mid \widehat{C}_j$. But $\widehat{C}_j$ is in the kernel of D , so $\vartheta^p \mid \widehat{C}_j$. Since ϑ is monic, we conclude that $\vartheta^{p(j-1)} C_j \in \mathbb{F}_p[\mathbf{a}, \mathbf{r}, x^p]$. The calculation at ∞ is similar. This proves the theorem for distinct a_i . Now let $a'_1, \dots, a'_m$ be algebraically independent over $\mathbb{F}_p$. Writing $\vartheta(\mathbf{a}', x) = \prod_{i=1}^n (x - a'_i)$ we have that $\widehat{C}_j(\mathbf{a}', \mathbf{r}, x^p) = \vartheta(\mathbf{a}', x)^p \widetilde{C}_j(\mathbf{a}', \mathbf{r}, x^p)$ as a formal relation in $\mathbb{F}_p[\mathbf{a}', \mathbf{r}, x^p]$ for some $\widetilde{C}_j(\mathbf{a}', \mathbf{r}, x^p)$ in that ring. Such a relation may be specialized $\mathbf{a}' \mapsto \mathbf{a}$ giving the result for arbitrary $\mathbf{a} \in \mathfrak{F}^p$.

Corollary 7. *Let $m, n \geq 2$ and let $\mathbf{s} \in S(m, n)$. There are $N = (n-1) \times (n(m-1)-2)/2$ conditions for the accessory parameters, i.e., there exist N polynomials $H_i(\mathbf{a}, \mathbf{r}) \in \mathbb{F}_p[\mathbf{a}, \mathbf{r}]$ and $L = L_{\mathbf{a}, \mathbf{r}, \mathbf{s}}$ has nilpotent p -curvature if and only if $H_i(\mathbf{a}, \mathbf{r}) = 0$, $i = 1, \dots, N$, i.e., $(\mathbf{a}, \mathbf{r}) \in W(m, n, \mathbf{s})$ if and only if $H_i(\mathbf{a}, \mathbf{r}) = 0$, $i = 1, \dots, N$. The polynomials H_i depend on $\mathbf{s} = (s_{ij})$.*

Proof. Since $\widetilde{C}_j = \vartheta^{p(j-1)} C_j \in \mathbb{F}_p[\mathbf{a}, \mathbf{r}, x^p]$ and $\deg_x \widetilde{C}_j < p \cdot k(j)$, $\deg_{x^p} \widetilde{C}_j < k(j)$. If we now equate all the coefficients of $(x^p)^l$ to 0 then we get at most

$\sum_{j=2}^n k(j) = N$ conditions for the accessory parameters ($N =$ number of accessory parameters).

Remark. Some of the conditions $H_i(\mathbf{a}, \mathbf{r}) = 0$ might be redundant or identically zero, but it follows from Theorem 9 that this is not the case. The corollary implies that $\text{trace}(G_p) = 0$, and therefore the trace gives no condition at all.

3. PROOF OF THE NONEMPTYNESS

3.1. The general case $n \geq 2, m \geq 2$. Let the operator L be parametrized as in §1, and let the varieties $V = V(m, n, \mathbf{s})$ and $W = W(m, n, \mathbf{s})$ be defined as in §1. Let $n \geq 2$ and $m \geq 2$. Then we have

Theorem 8. *The variety $W(m, n, \mathbf{s})$ is nonempty.*

Proof. We show that if $m, n \geq 2$ and $\mathbf{s} \in S(m, n)$ then $W(m, n, \mathbf{s})$ is nonempty, assuming that $W(m-1, n, \tilde{\mathbf{s}})$ is nonempty for all $\tilde{\mathbf{s}} \in S(m-1, n)$. We start with a given $\mathbf{s} \in S(m, n)$. By hypothesis, $\mathbf{s}$ satisfies Fuchs's relation

$$\sum_{i=1}^{m+1} \sum_{j=1}^n s_{ij} = (m-1) \binom{n}{2}.$$

We set $\tilde{m} = m-1$ and define $\tilde{\mathbf{s}} \in S(m-1, n)$ as follows:

- $m \geq 3$: (i) $\tilde{s}_{m-1,j} = s_{m-1,j} + s_{m,j} + (j-1)$,
- (ii) $\tilde{s}_{i,j} = s_{i,j}$,
- (iii) $\tilde{s}_{\tilde{m}+1,j} = s_{m+1,j}$, $i = 1, \dots, m-2$, $j = 1, \dots, n$
- ((i) could be replaced by the weaker requirement
- $\sum_{j=1}^n \tilde{s}_{m-1,j} = \sum_{j=1}^n s_{m-1,j} + \sum_{j=1}^n s_{m,j} + ((\frac{n}{2})))$;
- $m = 2$: (i) $\tilde{s}_{1,j} = -s_{m+1,j}$,
- (ii) $\tilde{s}_{\tilde{m}+1,j} = s_{m+1,j}$, $j = 1, \dots, n$.

The important points in the construction of $\tilde{\mathbf{s}}$ are (i) $\tilde{c}_{m-1,1} = c_{m-1,1} + c_{m,1}$; (ii) $\tilde{c}_{ij} = c_{ij}$, $1 \leq i \leq m-2$, $1 \leq j \leq n$; and (iii) $\sum_{i=1}^{m-1} \tilde{c}_{ij} + \tilde{c}_{\tilde{m}+1,j} = \sum_{i=1}^m c_{ij} + c_{m+1,j}$. Clearly, $\tilde{\mathbf{s}}$ satisfies the Fuchs relation $\sum_{i=1}^{\tilde{m}+1} \sum_{j=1}^n \tilde{s}_{ij} = (\tilde{m}-1) \binom{n}{2}$. Therefore, for $m \geq 2$ there exists by the induction hypothesis $(\tilde{\mathbf{a}}, \tilde{\mathbf{r}}) \in W(m-1, n, \tilde{\mathbf{s}})$ and an operator with nilpotent p -curvature $L = L_{\tilde{\mathbf{a}}, \tilde{\mathbf{r}}, \tilde{\mathbf{s}}}$, where $\tilde{\mathbf{a}} = (a_1, \dots, a_{m-1})$. If $m = 2$ then the unique solution is

$$L = \frac{1}{(x-a_1)^n} \prod_{j=1}^n \left((x-a_1) \frac{d}{dx} - s_{1,j} \right).$$

We now rewrite the operator

$$L = L_{m-1} = D^n + \sum_{j=1}^n \left(\sum_{i=1}^{m-1} \frac{\tilde{c}_{ij}}{(x-a_i)^j} + \frac{\tilde{c}_{\tilde{m}+1,j} x^{\tilde{k}(j)} + \sum_{l=0}^{\tilde{k}(j)-1} \tilde{r}_l x^l}{\tilde{\delta}^{j-1}} \right) D^{n-j}$$

as an operator $L = L_m$ with fictitiously m singularities by setting $a_m = a_{m-1}$,

i.e., $\mathbf{a} = (a_1, \dots, a_{m-1}, a_m)$:

$$\begin{aligned} L_m &= D^n + \sum_{j=1}^n \left(\sum_{i=1}^m \frac{c_{ij}}{(x-a_i)^j} + \left(\sum_{i=1}^{m-1} \frac{\tilde{c}_{ij}}{(x-a_i)^j} + \frac{\tilde{c}_{m+1,j} x^{\tilde{k}(j)} + \sum_{l=0}^{\tilde{k}(j)-1} \tilde{r}_l x^l}{\tilde{\vartheta}^{j-1}} \right. \right. \\ &\quad \left. \left. - \sum_{i=1}^m \frac{c_{ij}}{(x-a_i)^j} \right) \right) D^{n-j} \\ &= D^n + \sum_{j=1}^n \left(\sum_{i=1}^m \frac{c_{ij}}{(x-a_i)^j} + \frac{c_{m+1,j} x^{k(j)} + \sum_{l=0}^{k(j)-1} r_l x^l}{\vartheta^{j-1}} \right) D^{n-j}. \end{aligned}$$

More explicitly, we assert that there is a unique $\mathbf{r} \in \mathfrak{F}^N$ such that $L_{\tilde{\mathbf{a}}, \tilde{\mathbf{r}}, \tilde{\mathbf{s}}} = L_{\mathbf{a}, \mathbf{r}, \mathbf{s}}$, i.e., such that

$$\begin{aligned} \sum_{i=1}^{m-1} \frac{\tilde{c}_{ij}}{(x-a_i)^j} + \frac{\tilde{c}_{m+1,j} x^{\tilde{k}(j)} + \sum_{l=0}^{\tilde{k}(j)-1} \tilde{r}_{j,l} x^l}{\tilde{\vartheta}^{j-1}} - \sum_{i=1}^m \frac{c_{ij}}{(x-a_i)^j} \\ = \frac{c_{m+1,j} x^{k(j)} + \sum_{l=0}^{k(j)-1} r_{j,l} x^l}{\vartheta^{j-1}} \end{aligned}$$

where $\vartheta(x) = (x - a_{m-1}) \cdot \tilde{\vartheta}(x)$ and $j = 1, \dots, n$. After multiplication by $\vartheta^{j-1} = (\tilde{\vartheta} \cdot (x - a_{m-1}))^{j-1}$ we get the polynomial equation

$$\begin{aligned} \sum_{i=1}^{m-1} \tilde{c}_{ij} \frac{\vartheta^{j-1}}{(x-a_i)^j} - \sum_{i=1}^m c_{ij} \frac{\vartheta^{j-1}}{(x-a_i)^j} + \left[\tilde{c}_{m+1,j} x^{\tilde{k}(j)} + \sum_{l=0}^{\tilde{k}(j)-1} \tilde{r}_{j,l} x^l \right] (x-a_{m-1})^{j-1} \\ = c_{m+1,j} x^{k(j)} + \sum_{j=0}^{k(j)-1} r_{j,l} x^l. \end{aligned}$$

The first two sums taken together form a polynomial because (i) for $i = 1, \dots, m-2$ we have $\tilde{c}_{ij} = c_{ij}$; (ii) for $i = m-1$, $i = m$, $j = 2, \dots, n$, the denominator divides the numerator since $a_{m-1} = a_m$; and (iii) $\tilde{c}_{m-1,1} = c_{m-1,1} + c_{m,1}$. Therefore, it is enough to check that the left side is of degree bounded by $k(j)$ and that the coefficients of $x^{k(j)}$ on the two sides coincide. The coefficients of $x^{k(j)}$ on the two sides do coincide since

$$\left(\sum_{i=1}^{m-1} \tilde{c}_{ij} - \sum_{i=1}^m c_{ij} + \tilde{c}_{m+1,j} \right) + \sum_{i=1}^m c_{ij} = c_{m+1,j} + \sum_{i=1}^m c_{ij}$$

is determined in both cases by the exponents at ∞ . Therefore, $L = L_{m-1} = L_m$ has nilpotent p -curvature, so $(\tilde{\mathbf{a}}, \tilde{\mathbf{r}}) \in W(m-1, n, \tilde{\mathbf{s}})$ implies $(\mathbf{a}, \mathbf{r}) \in W(m, n, \mathbf{s})$. This completes the proof.

Theorem 9. *The variety $W(m, n, \mathbf{s})$ is a complete intersection of dimension m .*

Proof. Corollary 7 implies that $\dim W(m, n, \mathbf{s})$ is either -1 or $\geq m$. The former case is excluded by Theorem 8. It follows from Theorem 3 that $\dim W(m, n, \mathbf{s}) \leq m$. Therefore, the number of the defining equations for $W(m, n, \mathbf{s})$ is equal to the codimension of $W(m, n, \mathbf{s})$.

Remark. We fixed only the singularity at ∞ . In fact, we are free to fix three singularities $(0, 1, \infty)$ and in this case $\dim W = m - 2$.

As a corollary to Theorem 4 and Theorem 9 we obtain the following theorem which was previously demonstrated for $n = 2$ in [Dw].

Theorem 10. *The set $V(m, n, \mathbf{s})$ is nonempty and for fixed $\mathbf{a}$ $V(m, n, \mathbf{s})$ is nonempty and finite.*

3.2. Direct proof for $n = 2$. We simplify the notation in this section:

$$\begin{aligned} L &= D^2 + \mathfrak{v}D + \mathfrak{u}, \\ D^l &\equiv \mathfrak{A}_l D + \mathfrak{B}_l \pmod{\mathfrak{A}L}, \\ \mathfrak{z}_{l+1} &= \begin{pmatrix} \mathfrak{B}_{l+1} \\ \mathfrak{A}_{l+1} \end{pmatrix} = \begin{pmatrix} \mathfrak{B}_l \\ \mathfrak{A}_l \end{pmatrix}' + \begin{pmatrix} 0 & -\mathfrak{u} \\ 1 & -\mathfrak{v} \end{pmatrix} \begin{pmatrix} \mathfrak{B}_l \\ \mathfrak{A}_l \end{pmatrix} = (D + G)\mathfrak{z}_l. \end{aligned}$$

To compute the degree in $\mathbf{r}$ of $\mathfrak{z}_l$ we have to know the degree in $\mathfrak{u}$, $\mathfrak{u}'$, $\mathfrak{u}''$, ... (denoted by $\deg_{\mathfrak{u}}$) of $\mathfrak{z}_l$. From

$$\mathfrak{z}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathfrak{z}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathfrak{z}_{l+2} = (D + G)^2 \mathfrak{z}_l$$

we obtain inductively:

$$\begin{aligned} l \text{ odd: } \deg_{\mathfrak{u}} \mathfrak{A}_l &= \frac{l-1}{2}, \quad \deg_{\mathfrak{u}}(\mathfrak{A}_l - (-\mathfrak{u})^{(l-1)/2}) < \frac{l-1}{2}, \\ \deg_{\mathfrak{u}} \mathfrak{B}_l &\leq \frac{l-1}{2}; \\ l \text{ even: } \deg_{\mathfrak{u}} \mathfrak{A}_l &\leq \frac{l-2}{2}, \\ \deg_{\mathfrak{u}} \mathfrak{B}_l &= \frac{l}{2}, \quad \deg_{\mathfrak{u}}(\mathfrak{B}_l - (-\mathfrak{u})^{l/2}) < \frac{l}{2}. \end{aligned}$$

Hence, if

$$\Delta = C_2 = \begin{vmatrix} \mathfrak{B}_p & \mathfrak{B}_{p+1} \\ \mathfrak{A}_p & \mathfrak{A}_{p+1} \end{vmatrix},$$

then $\deg_{\mathfrak{u}} \Delta = p$ and $\deg_{\mathfrak{u}}(\Delta + (-\mathfrak{u})^p) < p$. Using

$$\mathfrak{u} = \sum_{i=1}^m \frac{c_{i,2}}{(x-a_i)^2} + \frac{c_{m+1,2}x^{m-2} + \sum_{i=0}^{m-3} r_i x^i}{\vartheta},$$

we obtain that the leading coefficient of $\vartheta^{2p} \Delta = \tilde{C}_2$ relative to $\mathbf{r}$ is $\vartheta^p \sum r_i^p x^{ip}$ and the leading coefficient of $\tilde{C}_2$ relative to $\mathbf{r}$ is $r_i^p x^i$. This implies that the coefficient of x^i in $\tilde{C}_2$ is $r_i^p - \pi_i(\mathbf{a}, \mathbf{r})$, $\pi_i \in \mathbf{F}_p[\mathbf{a}, \mathbf{r}]$, $\deg_{\mathbf{r}} \pi_i < p$. Thus we get the $m-2$ equations

$$r_i^p = \pi_i(\mathbf{a}, \mathbf{r}), \quad 0 \leq i \leq m-3, \quad \deg_{\mathbf{r}} \pi_i < p,$$

to be satisfied by the r_i . The solution space of this system of equations is nonempty. To prove this we show that $1 \notin \sum_{i=0}^{m-3} (r_i^p - \pi_i) \mathbf{F}_p(\mathbf{a})[\mathbf{r}]$. Suppose otherwise. Then $1 = \sum (r_i^p - \pi_i) A_i$, $A_i \in \mathbf{F}_p(\mathbf{a})[\mathbf{r}]$. Let $d = \sup_i \deg_{\mathbf{r}} A_i$. We may assume this is a representation of 1 with d minimal. Let $A_{i,d}$ be the form of $\deg d$ in $\mathbf{r}$ in A_i . Then by the hypothesis for π_i , $0 = \sum r_i^p A_{i,d}$ and hence $\exists \{\eta_{ij}\}_{0 \leq i, j \leq m-3}$ such that $\eta_{ij} = -\eta_{ji}$, η_{ij} a form of $\deg d - p$ in $\mathbf{r}$ in $\mathbf{F}_p(\mathbf{a})[\mathbf{r}]$ such that $A_{i,d} = \sum_j \eta_{ij} r_j^p$. Let $A'_i = A_i - \sum \eta_{ij} (r_j^p - \pi_j)$. Then $1 = \sum A'_i (r_i^p - \pi_i)$ and $\deg_{\mathbf{r}} A'_i < d$, contradicting the minimality of d . If $\mathbf{a}$ is fixed then the number of solutions is bounded by p^{m-2} .

Conjecture. *By refining the counting argument in this section it should be possible to give a direct proof of the nonemptiness of $V(m, n, \mathbf{s})$ for any $m, n \geq 2$. Moreover, for fixed $\mathbf{a}$ the cardinality of the finite set $V(m, n, \mathbf{s})$ is bounded by p^N , where N is equal to the number of accessory parameters.*

Note added in proof. The conjecture has been proved by the author since the submission of this paper.

CONCLUSION

The set $V(m, n, \mathbf{s})$ is nonempty for any $m, n \geq 2$ and for any set of exponents $s_{ij} \in \mathbb{F}_p$ satisfying Fuchs's relation. For any fixed set $\mathbf{a}$ of distinct singularities there is a finite (nonempty) collection of operators $L_{\mathbf{a}, \mathbf{s}}$ with the prescribed Riemann scheme. Furthermore, $V(m, n, \mathbf{s})$ is a subset of $W(m, n, \mathbf{s})$ which is a complete intersection of dimension m .

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