THE UNICELLULARITY OF CONTRACTIONS OF CLASS C_0

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. In this paper, we shall generalize the unicellularity of operators on finite-dimensional spaces to that of the contraction of class C_0 on Hilbert spaces.

We prove:

- (1) Each nilpotent operator on Hilbert space is Banach reducible (Theorem 3).
- (2) A contraction T of class C_0 on Hilbert space is unicellular if and only if T has one-point spectrum and every invariant subspace for T is cyclic (Theorem 6).
- (3) A contraction T of class C_0 on Hilbert space is unicellular if and only if T has one-point spectrum and all invariant subspaces of T are hyperinvariant subspaces of T (Theorem 8).

The aim of this paper is to discuss the unicellularity of contractions of class C_0 . We shall generalize the unicellularity of operators acting on finite-dimensional spaces to that on infinite-dimensional spaces.

Throughout this paper, we denote by H a complex, separable, and infinite-dimensional Hilbert space. The related concepts and symbols can be found in [4].

Definition 1. Let H_1 be a nontrivial invariant subspace of T . H_1 is called a Banach reducible subspace of T if there exists another nontrivial invariant subspace H_2 of T such that

$$H = H_1 \dot{+} H_2,$$

where the symbol $\dot{+}$ denotes the direct sum.

Theorem 1. Let T be a contraction of class C_0 on H . Assume that its spectrum $\sigma(T)$ has more than one point. Then T has a Banach reducible subspace. Moreover, T is not a unicellular operator.

Proof. We consider two cases. First suppose that there exists at least one point $a \in \sigma(T) \cap D$, $|a| < 1$, where we denote by D the open unit disk. In accordance with the method of [4], we could show that there exist two invariant subspaces H_1, H_2 of T such that $H = H_1 \dot{+} H_2$. Hence T is not unicellular.

Received by the editors March 2, 1992.

1991 *Mathematics Subject Classification.* Primary 47A15, 47B99.

Project supported by the National Natural Science Foundation of China.

Secondly, suppose that $\sigma(T) \subseteq C$, where we denote the unit circle by C . Let $a, b \in \sigma(T)$, $a \neq b$. Take an arc $S \subseteq C$ such that $a \in S$, $b \in S^c$, where S^c denotes the remaining arc of S about C and such that the minimal function of T is

$$m_T(\lambda) = \chi S(\lambda), \quad |\lambda| < 1, \quad |\chi| = 1.$$

$S(\lambda)$ is a singular function

$$S(\lambda) = \exp \left(- \int_0^{2\pi} \frac{e^{it} + \lambda}{e^{it} - \lambda} d\mu_t(\lambda) \right), \quad |\lambda| < 1,$$

where $\mu_t(\lambda)$ is a positive measure on Lebesgue measurable sets. We put

$$S_1(\lambda) = \exp \left(- \int_{(S)} \frac{e^{it} + \lambda}{e^{it} - \lambda} d\mu_t(\lambda) \right), \quad |\lambda| < 1,$$

$$S_2(\lambda) = \exp \left(- \int_{(S^c)} \frac{e^{it} + \lambda}{e^{it} - \lambda} d\mu_t(\lambda) \right), \quad |\lambda| < 1,$$

where we denote by (S) the set $\{t; t \in [0, 2\pi), e^{it} \in S\}$ and by (S^c) the set $\{t; t \in [0, 2\pi), e^{it} \in S^c\}$. By [4], $S_1(\lambda)$, $S_2(\lambda)$ are the inner factors of $m_T(\lambda)$. Since $S \cap \sigma(T) \neq \emptyset$, $S^c \cap \sigma(T) \neq \emptyset$, and $S \cap \sigma(T) \neq \sigma(T)$, we have

$$0 < |S_1(0)| < \exp(-\mu_t(S)) < 1, \quad 0 < |S_2(0)| < \exp(-\mu_t(S^c)) < 1.$$

Thus $S_1(\lambda)$, $S_2(\lambda)$ are nontrivial inner factors of $m_T(\lambda)$. Put

$$m_1(\lambda) = \chi S_1, \quad m_2(\lambda) = S_2(\lambda), \quad |\lambda| < 1.$$

Again set

$$u_1(\lambda) = c_1 \exp \left(- \int_{(S)} \frac{e^{it} + \lambda}{e^{it} - \lambda} d\mu_t(\lambda) \right), \quad |\lambda| < 1,$$

$$u_2(\lambda) = c_2 \exp \left(- \int_{(S^c)} \frac{e^{it} + \lambda}{e^{it} - \lambda} d\mu_t(\lambda) \right), \quad |\lambda| < 1.$$

By [3], $u_1, u_2 \in H^\infty$ and $u_1(\lambda) \neq 0$, $u_2(\lambda) \neq 0$, $|\lambda| < 1$. Let us take c_1, c_2 such that

$$m_1(\lambda)u_1(\lambda) + m_2(\lambda)u_2(\lambda) = 1, \quad |\lambda| < 1.$$

By [4, Proposition III.6.4],

$$H = H_1 \dot{+} H_2,$$

where

$$H_1 = \{h; h \in H, m_1(T)h = 0\}, \quad H_2 = \{h; h \in H, m_2(T)h = 0\}.$$

Clearly, H_1, H_2 are invariant subspaces of T , and since $m_1(\lambda), m_2(\lambda)$ are nontrivial inner factors of $m_T(\lambda)$, we see that H_1, H_2 are nontrivial invariant subspaces of T . So T is not of unicellularity.

Theorem 2. Let T be an algebra operator on H . Then T has a Banach reducible subspace, hence, T is nonunicellular.

Proof. If the spectrum $\sigma(T)$ of T has more than one point, by Riesz-Dunford functional calculus, T has a Banach reducible subspace; hence, T is nonunicellular.

If the spectrum $\sigma(T)$ of T consists of a single point a , let $T_1 = aI - T$. Then $\sigma(T_1) = \{0\}$ and consequently T_1 is a nilpotent operator.

We may assume its degree is p , thus, we only need to show the following theorem:

Theorem 3. *Let T be a nilpotent operator of degree p . Then T has a Banach reducible subspace; furthermore, T is nonunicellular.*

Proof. We put

$$x_1, x_2 = Tx_1, x_3 = T^2x_1, \dots, x_p = T^{p-1}x_1,$$

where $x_1 \in H$, $T^{p-1}x_1 \neq 0$. Obviously, $x_1, x_2, \dots, x_p$ are linear independent elements of H . Let H_1 be the closed linear subspace of H spanned by the elements $x_1, x_2, \dots, x_p$.

It remains to prove that H_1 is a Banach reducible subspace of T . If $\{y_j; \}_{j=1}^p \subseteq H$ such that $\{x_i\}_{i=1}^p$ and $\{y_j\}_{j=1}^p$ are a biorthogonal system, then from the equation

$$Px = \sum_{i=1}^p (x, y_i)x_i, \quad x \in H,$$

we can determine a projection $P: H \rightarrow H_1$. Because $T^ix_1 = x_{i+1}$, $i = 1, 2, \dots, p-1$, it is easy to see that

$$TPx = \sum_{i=1}^p (x, y_i)Tx_i = \sum_{i=1}^{p-1} (x, y_i)x_{i+1}, \quad x \in H.$$

On the other hand, if we put $y_j = T^{*p-j}y$, $j = 1, 2, \dots, p$, i.e.,

$$y_1 = T^{*p-1}y, y_2 = T^{*p-2}y, \dots, y_{p-1} = T^*y, y_p = y,$$

then $T^*y_1 = 0$, $T^*y_{j+1} = y_j$; $j = 1, 2, \dots, p-1$, thus, we have

$$\begin{aligned} PTx &= \sum_{i=1}^p (Tx, y_i)x_i = \sum_{i=1}^p (x, T^*y_i)x_i \\ &= \sum_{i=2}^p (x, y_{i-1})x_i = \sum_{i=1}^{p-1} (x, y_i)x_i, \quad x \in H. \end{aligned}$$

It follows that $TP = PT$.

Now we shall prove that there exists an element $y \in H$ such that

$$(y_j, x_i) = (T^{*p-j}y, x_i) = (y, T^{p-j}x_i) = (y, x_{p+i-j}) = \delta_{ij}.$$

Clearly,

$$T^{p-j}x_i = \begin{cases} x_{p+(i-j)} = x_p, & i = j, \\ x_{p+(i-j)}, & i = 1, 2, \dots, j-1, \\ 0, & i = j+1, \dots, p; \end{cases}$$

therefore, there is a $y \in H$ such that

$$(y, x_i) = \begin{cases} 1, & i = p, \\ 0, & i = 1, 2, \dots, p-1. \end{cases}$$

The proof of Theorem 3 is complete.

Theorem 3 obviously improves the results of Apostol and Voiculescu [1].

Corollary 4. *Let T be an operator of class C_0 and $\|T\| < 1$. Then T has a Banach reducible subspace; further, T is nonunicellular.*

Proof. Since $\|T\| < 1$, the minimal function $m_T(\lambda)$ of T is an analytic function on the closed unit disk D ; thus, $m_T(\lambda)$ is a rational inner function and hence T is an algebra operator by Theorem 2. Therefore Corollary 4 is true.

By Corollary 4 and the unicellularity of contraction of class C_0 , we only need to discuss the case of $\|T\| = 1$ with one-point spectrum, i.e.,

$$\sigma(T) = \{a\}, \quad |a| = 1.$$

In this case, T is unicellular, for example, $\sigma(T) = \{1\}$, whose defect indices are equal to 1 (cf. [4]). T also has Banach reducible subspace, i.e., T is not unicellular, for example, $H = H \oplus H$, $T = T \oplus T$, where T is an operator of class C_0 , whose spectrum only consists of a single point a , $\|T\| = 1$, $|a| = 1$.

Next we suppose T is a contraction of class C_0 whose spectrum consists of a single point, and $\|T\| = 1$. Our aim is to establish the principia of the unicellularity for T . This has been investigated in reference [7]. We shall investigate the relations between the one-spectrum and the lattice of invariant subspaces for T .

Theorem 5. *Let T be a contraction of class C_0 on H . Then T is unicellular if and only if T has one-point spectrum and T does not have two different invariant subspaces H_1, H_2 , such that $T|_{H_1}$ and $T|_{H_2}$ have the same minimal function.*

Proof. Let T be unicellular of class C_0 by [4, Corollary III.7.3], $\sigma(T) = \{a\}$, $|a| \leq 1$. By Corollary 4 we have $|a| = 1$; hence, the minimal function of T is

$$m_T(\lambda) = \exp\left(a_T \frac{\lambda + a}{\lambda - a}\right), \quad a_T > 0.$$

By [4, Proposition III.7.5] $\text{Lat } T = \{H_s\}_{0 \leq s \leq a_T}$, where

$$H_s = \{x : x \in H, e_s(T)x = 0\},$$

and

$$e_s(\lambda) = \exp\left\{s \frac{\lambda + a}{\lambda - a}\right\}, \quad 0 \leq s \leq a_T.$$

If $H_1, H_2 \in \text{Lat } T$, $H_1 \neq H_2$, then there exist two positive numbers s_1, s_2 such that $s_1 \neq s_2$ and $H_1 = H_{s_1}$, $H_2 = H_{s_2}$. Thus for $T_1 = T|_{H_1}$, $T_2 = T|_{H_2}$, we have

$$m_{T_1}(\lambda) = \exp\left(s_1 \frac{\lambda + a}{\lambda - a}\right), \quad m_{T_2}(\lambda) = \exp\left(s_2 \frac{\lambda + a}{\lambda - a}\right),$$

which implies that $m_{T_1}(\lambda) \neq m_{T_2}(\lambda)$. Now we shall show the sufficiency of Theorem 5. Since T is a contraction of class C_0 and $\sigma(T) = \{a\}$, $|a| \leq 1$, the minimal function of T is

$$m_T(\lambda) = \begin{cases} \left(\frac{\lambda + a}{\lambda - a}\right)^s, & |a| < 1, \\ \exp\left(a_T \frac{\lambda + a}{\lambda - a}\right), & |a| = 1, a_T > 0. \end{cases}$$

Remark, for $|a| < 1$, $\text{Lat } T = \{H_v\}_{v=0}^s$, where

$$H_v = \left\{x : x \in H, \left(\frac{T + a}{T - a}\right)^v x = 0\right\}, \quad v = 0, 1, 2, \dots, n;$$

in fact, if $H' \neq H_v$, $v = 0, 1, 2, \dots, n$, by [4, Theorem III.6.3]

$$m_{T|H_v} \left(\frac{\lambda + a}{\lambda - a} \right)^v, \quad m_{T'}(\lambda) = m_{T|H'}(\lambda),$$

then

$$m_{T'}(\lambda) \neq m_{T|H_v}(\lambda), \quad v = 0, 1, 2, \dots, n.$$

Further we have

$$m_{T'}(\lambda) \neq \left(\frac{\lambda - a}{1 - \bar{a}\lambda} \right)^v, \quad v = 0, 1, 2, \dots, n.$$

On the other hand, by [4, Proposition III.6.1] $T' = T|H'$ is a contraction of class C_0 , too, and the minimal function $m_{T'}(\lambda)$ of T' is an inner factor of $m_T(\lambda)$, and since $m_T(\lambda) = [(\lambda - a)/(1 - \bar{a}\lambda)]^n$, hence there is v' , $0 \leq v' \leq n$, such that

$$m_{T'}(\lambda) = \left(\frac{\lambda - a}{1 - \bar{a}\lambda} \right)^{v'}.$$

This is a contradiction; consequently, there must exist v_0 , $0 \leq v_0 \leq n$, such that $H' = H_{v_0}$. Thus $H' \in \{H_v\}_{v=0}^n = 0$, i.e.,

$$\text{Lat } T = \{H_v\}_{v=0}^n.$$

It follows that T is unicellular.

For $|a| = 1$, $m_T(\lambda) = \exp[a_T(\lambda + a)/(\lambda - a)]$, the proof can be completed similarly; thus, we omit it.

The following theorem generalizes the result of unicellularity in finite-dimensional spaces.

Theorem 6. *Let T be a contraction of class C_0 on H . Then T is unicellular if and only if T has one-point spectrum and each invariant subspace for T is cyclic.*

Proof. By [4, Corollary III.7.3] T has one-point spectrum.

Again by [6, Corollary 4.5] the invariant subspaces of T are cyclic.

Conversely, if there exist two invariant subspaces H_1 and H_2 of T such that the minimal functions of $T_1 = T|H_1$ and $T_2 = T|H_2$ are the same minimal function, then

$$m_{T_1}(\lambda) = m_{T_2}(\lambda), \quad |\lambda| < 1.$$

Since the invariant subspaces of T are cyclic so that T has a cyclic vector, by [5, Theorem 2] there exists an invariant subspace H_0 of T ,

$$H_0 = \{x : x \in H, m_{T_1}(T)x = 0\},$$

such that

$$m_{T_1}(\lambda) = m_{T|H_0}(\lambda), \quad |\lambda| < 1.$$

Hence

$$m_{T_2}(\lambda) = m_{T|H_0}(\lambda), \quad |\lambda| < 1.$$

Obviously, for all $x \in H_1$ or $x \in H_2$ we have

$$m_{T|H_0}(T)x = m_{T_1}(T)x = m_{T_2}(T)x = 0.$$

It follows that $x \in H$; therefore, $H_1 \subseteq H_0$, $H_2 \subseteq H_0$. Since $H_1 \neq H_2$, it is easy to see that there exists H_1 or H_2 such that it is different from H_0 . Now

we assume that $H_1 \neq H_0$. Because $T|H_0$ is a contraction of class C_0 and H_0 is a cyclic subspace of T , it follows that $T|H_0$ has also a cyclic vector. As above, there is an invariant subspace H' of $T|H_0$ and $H' \neq H$ such that the minimal function of $T' = (T|H_0)|_{H'} = T|_{H'}$ is

$$m_{T'}(\lambda) = m_{T'|H_0}(\lambda) = m_{T|H_0|H'}(\lambda) = m_{T|H_0}.$$

By [5, Theorem 2] this is impossible. Hence there exist no two different invariant subspaces H_1, H_2 of T such that $T|H_1, T|H_2$ have the same minimal functions.

From Theorem 5, T is unicellular.

Theorem 7. *Let T be a contraction of class C_0 on H . Then T is unicellular if and only if there exists no Banach irreducible subspace of T and all invariant subspaces of T are cyclic.*

Proof. By Theorem 2, T has one-point spectrum; by Theorem 6, T is unicellular. Conversely, by Theorem 6, T has one-point spectrum and all invariant subspaces of T are cyclic subspaces. It remains to prove that T has no Banach reducible subspace.

Now, let us suppose that there exist invariant subspaces H_1 and H_2 of T such that

$$H = H_1 \dot{+} H_2;$$

note that $T_i = T|_{H_i}$ ($i = 1, 2$) also are contractions of class C_0 and

$$\sigma(T_i) = \sigma(T) = \{a\}, \quad |a| \leq 1 \quad (i = 1, 2).$$

By Corollary 4, $|a| = 1$. From the definition of the minimal function in reference [1], we have

$$m_T(\lambda) = \exp \left\{ a_T \frac{\lambda + a}{\lambda - a} \right\}, \quad |a| = 1, \quad a_T > 0.$$

Again from [4, Proposition III.6.1] $m_{T_i}(\lambda)$ is an inner factor of the minimal function $m_T(\lambda)$; hence, there exists a positive number a_i , $0 < a_i < a$, such that

$$m_{T_i}(\lambda) = \exp \left(a_i \frac{\lambda + a}{\lambda - a} \right), \quad i = 1, 2, \quad |a| = 1.$$

Without loss of generality, we may assume that $0 < a_1 < a_2 < a$. Let $g(\lambda) \in H^\infty$ such that

$$m_{T_2}(\lambda) = g(\lambda)m_{T_1}(\lambda), \quad |\lambda| < 1.$$

Denote the projection from H onto H_1 by P_1 ; thus, $P_1T = TP_1$ and consequently, for every $x \in H$, we have $P_1x \in H_1$. Hence

$$0 = m_{T_1}(T_1)P_1x = P_1m_{T_1}(T)x, \quad x \in H.$$

This implies that

$$m_{T_1}(T)x \in H_2, \quad x \in H.$$

Therefore

$$m_{T_1}(T)H \subseteq H_2.$$

Similarly as above, we have

$$m_{T_2}(T)H \subseteq H_1.$$

Since

$$m_{T_2}(\lambda) = g(\lambda)m_{T_1}(\lambda), \quad |\lambda| < 1,$$

for every $x \in H$

$$m_{T_2}x = g(T)m_{T_1}(T)x \in g(T)H_2 \subseteq H_2.$$

It follows that

$$m_{T_2}(T)H \subseteq H_2.$$

Consequently, we obtain

$$m_{T_2}(T)H \subseteq H_1 \cap H_2 = \{0\},$$

which implies that

$$m_{T_2}(T) = 0.$$

Since $m_T(\lambda)$ is a minimal function for T , $m_{T_2}(\lambda)$ is an inner factor of $m_T(\lambda)$; therefore, $m_{T_2}(\lambda) = m_T(\lambda)$. By [5, Theorem 2] T does not have a cyclic vector, which is a contradiction. Hence there is no Banach reducible subspace for T .

Obviously the following result is a generalization of the result of the uncellularity on finite-dimensional spaces.

Theorem 8. *Let T be a contraction of class C_0 on H . Then T is unicellular if and only if T has a one-point spectrum and all invariant subspaces of T are hyperinvariant subspaces of T .*

Proof. The “only if” part follows from [4, Propositions III.7.3, II.7.6].

Next let H_1 be an invariant subspace for T , $T_1 = T|_{H_1}$; hence,

$$\text{Lat } T_1 = \{M \cap H_1; M \in \text{Lat } T\}.$$

Clearly, the invariant subspaces of T_1 are the hyperinvariant subspaces and T_1 is of one-point spectrum of class C_0 , too.

Assume H_2 is an invariant subspace of T_1 , $T_2 = T_1|_{H_2} = T|_{H_2}$, and H_2 is also a hyperinvariant subspace of T_1 . If $m_{T_2}(\lambda) = m_{T_1}(\lambda)$, then

$$H_1 = H_2 \oplus H_3, \quad T_1 = \begin{bmatrix} T_2 & X \\ 0 & T_3 \end{bmatrix}$$

where T_3 is a composition of T_1 on H_3 , i.e., $T_3 = PT_1|_{H_3}$ where P is a projection from H_1 onto H_3 . Since the invariant subspaces of T_1 are hyperinvariant, by [2, Corollary 2.6] we have

$$m_{T_1}(\lambda) = m_{T_2}(\lambda)m_{T_3}(\lambda) = m_{T_1}(\lambda), \quad |\lambda| < 1.$$

This implies that $m_{T_2}(\lambda) = 1$ admits a scalar multiple of modulus 1. Hence

$$m_{T_3}(T_1) = I|_{H_3}.$$

Consequently

$$\begin{aligned} I|_{H_3} &= PI|_{H_3} = P_{m_{T_3}}(T_1)|_{H_3} = m_{T_3}(P \cdot T_1)|_{H_3} \\ &= m_{T_3}(T_3)|_{H_3} = 0|_{H_3}. \end{aligned}$$

Thus we must have that $H_3 = \{0\}$ so that we obtain the contradiction. Therefore there exists a nontrivial invariant subspace H_2 for T_1 such that

$$m_{T_2}(\lambda) = m_{T_1}(\lambda), \quad |\lambda| < 1.$$

By [5, Theorem 2] $T_1 = T|_{H_1}$ has a cyclic vector. Further, H_1 is a cyclic subspace of T_1 , and from Theorem 4 it follows that T is unicellular. Moreover, by Theorems 6 and 7 we can deduce Theorem 9: *Let T be a contraction of class C_0 on H . Then T is unicellular if and only if there exists no Banach reducible subspace of T and all invariant subspaces of T are hyperinvariant subspaces of T .*

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