NORMALITY OF σ -PRODUCTS

KEIKO CHIBA

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Dedicated to Professor Akihiro Okuyama on his 60th birthday

ABSTRACT. We prove that no σ -product of an uncountable family of nontrivial spaces is hereditarily normal or hereditarily ω_1 -paracompact. Also, if every finite subproduct of a σ -product is (hereditarily) countably paracompact and (hereditarily) normal, then the σ -product is (hereditarily) countably paracompact iff it is (hereditarily) normal.

1. INTRODUCTION

We later define σ -products which were introduced by Corson [3].

Throughout this paper we assume that κ is an infinite cardinal, $\mathcal{S} = \{X_\alpha | \alpha < \kappa\}$ where each X_α is a Hausdorff space having at least two points, and σ is a σ -product of $\mathcal{S}$.

Kombarov proved:

(A) (Kombarov [4, Theorem 3]). *If every finite subproduct of σ is paracompact, then σ is paracompact.*

Further he essentially showed:

(B) *If σ is normal and λ is an infinite cardinal and every finite subproduct of σ is λ -paracompact, then σ is λ -paracompact.*

Here $\prod_{\alpha \in F} X_\alpha$ is said to be a finite subproduct of σ if F is a finite subset of κ .

We prove:

Theorem 1. *If $\mathcal{S}$ is uncountable, σ is not hereditarily normal.*

Theorem 2. *If $\mathcal{S}$ is uncountable, σ is not hereditarily ω_1 -paracompact.*

Theorem 3. *If $\mathcal{S}$ is countable and every finite subproduct of σ is countably paracompact and normal, then σ is countably paracompact iff σ is normal.*

Theorem 4. *If $\mathcal{S}$ is countable and every finite subproduct of σ is hereditarily countably paracompact and hereditarily normal, then σ is hereditarily countably paracompact iff σ is hereditarily normal.*

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2. PRELIMINARIES

By definition, “ σ is a σ -product of $\mathcal{S}$ ” means there is a point $x^* \in X = \prod_{\alpha < \kappa} X_\alpha$ (called the base point of σ) such that σ is the subspace of X consisting of $\{x \in X \mid \{\alpha \in \kappa \mid x(\alpha) \neq x^*(\alpha)\} \text{ is finite}\}$.

We use standard set-theoretic notation including, for a set B , $B^{<\omega} = \{\text{finite subset of } B\}$ and, for $n \in \omega$, $B^n = \{\text{subset of } B \text{ of cardinality } n\}$. If $B \subset A \subset X$, $\text{cl } B$ means the closure of B in X while $\text{cl}_A B$ means the closure of B in A .

A space is called λ -paracompact [5] for some infinite cardinal λ if every open cover of the space of cardinality $\leq \lambda$ has a locally finite open refinement. It is (countably) metacompact [2] if every (countable) open cover has a point finite open refinement.

For $a \in \kappa^{<\omega}$, let $Y_a = \prod_{\alpha \in a} X_\alpha \times \prod_{\alpha \in \kappa - a} \{x^*(\alpha)\}$. Let $p_a: \sigma \rightarrow Y_a$ be the projection map defined by

$$p_a(x)(\alpha) = \begin{cases} x(\alpha) & \text{if } \alpha \in a, \\ x^*(\alpha) & \text{if } \alpha \in \kappa - a. \end{cases}$$

For $a, b \in \kappa^{<\omega}$, with $a \subset b$, define $p_a^b: Y_b \rightarrow Y_a$ by

$$p_a^b(x)(\alpha) = \begin{cases} x(\alpha) & \text{if } \alpha \in a, \\ x^*(\alpha) & \text{if } \alpha \in b - a. \end{cases}$$

Obviously p_a and p_a^b are continuous open maps and $p_a^b \circ p_b = p_a$ when $a \subset b$.

3. PROOF OF OUR THEOREMS

Proof of Theorem 1. Let $\mathcal{D} = \{2_\alpha \mid \alpha < \omega_1\}$, where, for each $\alpha < \omega_1$, 2_α is a copy of the discrete space $\{0, 1\}$ of two points. Let $d^* \in D = \prod_{\alpha < \omega_1} 2_\alpha$ be defined by $d^*(\alpha) = 0$ for all $\alpha < \omega_1$. If $\sigma(\mathcal{D})$ is the σ -product of $\mathcal{D}$ with base point d^* , then, for all uncountable $\mathcal{S}$, there is a homeomorphic embedding of $\sigma(\mathcal{D})$ in σ with x^* as the image of d^* . Thus it suffices to prove:

Proposition 1. $\sigma(\mathcal{D}) - \{d^*\}$ is not normal.

Proof. Let B and C be uncountable disjoint subsets of ω_1 . For $\alpha \in \omega_1$, let f_α be the point of D defined by $f_\alpha(\alpha) = 1$ but $f_\alpha(\beta) = 0$ for all $\beta \neq \alpha$. Let $E = \{f_\alpha \mid \alpha \in B\}$ and $F = \{f_\alpha \mid \alpha \in C\}$. Since d^* is the only limit point of E or F in D , E and F are closed and disjoint in $\sigma(\mathcal{D}) - \{d^*\}$. Suppose L is open in $\sigma(\mathcal{D}) - \{d^*\}$ and $E \subset L$. We find a point of F in the closure of L , thus proving that $\sigma(\mathcal{D}) - \{d^*\}$ is not normal.

For each $\alpha \in B$ there is a finite $a_\alpha \subset \omega_1 - \{\alpha\}$ such that $f_\alpha \in L_\alpha = \{f \in \sigma(\mathcal{D}) \mid f(\alpha) = 1 \text{ and } f(\beta) = 0 \text{ for all } \beta \in a_\alpha\} \subset L$. Since B is uncountable, B has an uncountable subset B' such that $\{a_\alpha \mid \alpha \in B'\}$ forms a Δ -system; that is, there is an $a \subset \omega_1$ such that $a_\alpha \cap a_\beta = a$ for all α and β in B' . Since a is finite, there is $\gamma \in C - a$; but $f_\gamma \in F$ and f_γ is a limit of $\bigcup_{\alpha \in B'} L_\alpha \subset L$, as desired.

Proposition 2. If every finite subproduct of σ is hereditarily normal and σ is hereditarily κ -paracompact, then σ is hereditarily normal.

Proof. Let $\mathcal{A} = \kappa^{<\omega}$. Let G be an arbitrary open subset of σ and $\mathcal{G} = \{G_1, G_2\}$ a binary open cover of G . To prove that σ is hereditarily normal, it

suffices to find a closed cover $\{K_1, K_2\}$ of G with $K_i \subset G_i$ for $i = 1, 2$. Let $U_{a,i} = \bigcup\{U|U \text{ is open in } Y_a, p_a^{-1}(U) \subset G_i\}$ for each $a \in \mathcal{A}$ and $i = 1, 2$, and put $U_a = \bigcup_{i=1}^2 U_{a,i}$. Then U_a is open in Y_a and $\{p_a^{-1}(U_a)|a \in \mathcal{A}\}$ is an open cover of G such that $p_a^{-1}(U_a) \subset p_b^{-1}(U_b)$ for each $a, b \in \mathcal{A}$ with $a \subset b$. Since $|\mathcal{A}| = \kappa$ and G is κ -paracompact, there is a locally finite open cover $\mathcal{J} = \{J_a|a \in \mathcal{A}\}$ of G such that $J_a \subset p_a^{-1}(U_a)$ for each $a \in \mathcal{A}$. Let $\Lambda = \mathcal{A}^{<\omega}$ and put $V_\lambda = G - \bigcup\{\text{cl}(J_b)|b \in \mathcal{A} - \lambda\}$ for each $\lambda \in \Lambda$. Then

- (1) $\mathcal{V} = \{V_\lambda|\lambda \in \Lambda\}$ is an open cover of G ;
- (2) if $\lambda, \mu \in \Lambda$ and $\lambda \subset \mu$, then $V_\lambda \subset V_\mu$; and
- (3) let $a_\lambda = \bigcup\{a|a \in \lambda\}$ for each $\lambda \in \Lambda$; then $a_\lambda \in \mathcal{A}$ and $\text{cl}_G V_\lambda \subset p_{a_\lambda}^{-1}(U_{a_\lambda})$.

Let $E_\lambda = \bigcup\{E|E \text{ is open in } Y_{a_\lambda}, p_{a_\lambda}^{-1}(E) \subset V_\lambda\}$ for each $\lambda \in \Lambda$. Then

- (4) $p_{a_\lambda}^{-1}(E_\lambda) \subset V_\lambda$;
- (5) $E_\lambda \subset U_{a_\lambda}$;
- (6) if $\lambda \subset \mu$, then $p_{a_\lambda}^{-1}(E_\lambda) \subset p_{a_\mu}^{-1}(E_\mu)$; and
- (7) $\{p_{a_\lambda}^{-1}(E_\lambda)|\lambda \in \Lambda\}$ is an open cover of G .

It is obvious that (4)–(6) hold.

Proof of (7). Let $x \in G$. Then $x \in V_\lambda$ for some $\lambda \in \Lambda$ by (1). Since V_λ is open in σ , there are an $a \in \mathcal{A}$ and an open set W in Y_a such that $x \in p_a^{-1}(W) \subset V_\lambda$. Let $\mu = \lambda \cup \{a\}$. Then $\mu \in \Lambda$ and $p_a^{-1}(W) = (p_a^{a_\mu} \circ p_{a_\mu})^{-1}(W) = p_{a_\mu}^{-1}((p_a^{a_\mu})^{-1}(W))$. Since $(p_a^{a_\mu})^{-1}(W)$ is open in Y_{a_μ} , $(p_a^{a_\mu})^{-1}(W) \subset E_\mu$ by the definition of E_μ . Thus $x \in p_{a_\mu}^{-1}(E_\mu)$.

Let $B_\lambda = \text{cl } E_\lambda \cap (\text{cl } U_{a_\lambda} - U_{a_\lambda})$ for each $\lambda \in \Lambda$. Then

- (8) $p_{a_\lambda}^{-1}(B_\lambda) \cap G = \emptyset$ for each $\lambda \in \Lambda$.

Proof of (8). Suppose there is an $x = (x_\alpha)_{\alpha < \kappa} \in p_{a_\lambda}^{-1}(B_\lambda) \cap G$. Then $p_{a_\lambda}(x) \in B_\lambda$. Thus $p_{a_\lambda}(x) \in \text{cl } E_\lambda$. Since $E_\lambda \subset U_{a_\lambda}$, $p_{a_\lambda}(x) \in \text{cl } U_{a_\lambda}$ and so $x \notin p_{a_\lambda}^{-1}(U_{a_\lambda})$. Since $\text{cl}_G(p_{a_\lambda}^{-1}(E_\lambda)) \subset \text{cl}_G V_\lambda \subset p_{a_\lambda}^{-1}(U_{a_\lambda})$, $x \notin \text{cl}_G(p_{a_\lambda}^{-1}(E_\lambda))$. On the other hand, we have $x \in \text{cl}_G(p_{a_\lambda}^{-1}(E_\lambda))$. This is a contradiction. To show that $x \in \text{cl}_G(p_{a_\lambda}^{-1}(E_\lambda))$, let H be an arbitrary open set such that $x \in H \subset G$. Then there are an $a \in \mathcal{A}$ and an open set W_a in Y_a such that $x \in p_a^{-1}(W_a) \subset H$. Let $\mu = \lambda \cup \{a\}$, and put $W_\mu = (p_a^{a_\mu})^{-1}(W_a)$. Then $x \in p_{a_\mu}^{-1}(W_\mu) \subset H$ and $p_{a_\lambda}^{a_\mu}(W_\mu)$ is a neighborhood of $p_{a_\lambda}(x)$ in Y_{a_λ} because $p_{a_\lambda}^{a_\mu}$ is an open map. Since $p_{a_\lambda}(x) \in \text{cl } E_\lambda$, $E_\lambda \cap p_{a_\lambda}^{a_\mu}(W_\mu) \neq \emptyset$. Let $b \in E_\lambda \cap p_{a_\lambda}^{a_\mu}(W_\mu)$. Then $b = p_{a_\lambda}^{a_\mu}(c)$ for some $c \in W_\mu$. Let $y \in \sigma$ be an element such that $p_{a_\mu}(y) = c$. Then $p_{a_\lambda}(y) = b$. Therefore, $y \in p_{a_\mu}^{-1}(W_\mu) \cap p_{a_\lambda}^{-1}(E_\lambda)$. Thus $H \cap p_{a_\lambda}^{-1}(E_\lambda) \neq \emptyset$.

Since $U_{a_\lambda} \cap \text{cl } E_\lambda = (\text{cl } E_\lambda) - B_\lambda \subset \bigcup_{i=1}^2 U_{a_\lambda, i}$, $(\text{cl } E_\lambda) - B_\lambda$ is closed in $Y_{a_\lambda} - B_\lambda$, and the $U_{a_\lambda, i}$ are open in Y_{a_λ} , there are closed sets $F_{a_\lambda, i}$ of $Y_{a_\lambda} - B_\lambda$ such that $F_{a_\lambda, i} \subset U_{a_\lambda, i}$ and $U_{a_\lambda} \cap \text{cl } E_\lambda = \bigcup_{i=1}^2 F_{a_\lambda, i}$. The $p_{a_\lambda}^{-1}(F_{a_\lambda, i})$ are closed sets of G because $p_{a_\lambda}^{-1}(F_{a_\lambda, i})$ are closed sets in $p_{a_\lambda}^{-1}(Y_{a_\lambda} - B_\lambda) = \sigma - p_{a_\lambda}^{-1}(B_\lambda)$ and $p_{a_\lambda}^{-1}(F_{a_\lambda, i}) \subset G \subset \sigma - p_{a_\lambda}^{-1}(B_\lambda)$.

Since $\{p_{a_\lambda}^{-1}(E_\lambda)|\lambda \in \Lambda\}$ is an open cover of G , $|\Lambda| = \kappa$, and G is κ -paracompact, there is a locally finite open cover $\{C_\lambda|\lambda \in \Lambda\}$ of G such that

$C_\lambda \subset p_{a_\lambda}^{-1}(E_\lambda)$ for each $\lambda \in \Lambda$. Let $K_i = \bigcup \{p_{a_\lambda}^{-1}(F_{a_\lambda, i}) \cap \text{cl}_G C_\lambda \mid \lambda \in \Lambda\}$ for $i = 1, 2$. Then K_i are closed sets in G such that $K_i \subset G_i$ and $G = \bigcup_{i=1}^2 K_i$.

Proof of Theorem 2. By Theorem 1, $\sigma(\mathcal{D})$ is not hereditarily normal and thus, by Proposition 2, $\sigma(\mathcal{D})$ is not hereditarily ω_1 -paracompact. Since every σ for uncountable $\mathcal{S}$ contains a copy of $\sigma(\mathcal{D})$, σ is not hereditarily ω_1 -paracompact.

Proposition 3. *If every finite subproduct of σ is normal and σ is κ -paracompact, then σ is normal.*

Proof. Let $\mathcal{G} = \{G_1, G_2\}$ be an arbitrary binary open cover of σ . Let $U_{a, i} = \bigcup \{U \mid U \text{ is open in } Y_a, p_a^{-1}(U) \subset G_i\}$ for each $a \in \mathcal{A}$ and $i = 1, 2$, and put $U_a = \bigcup_{i=1}^2 U_{a, i}$. Let us define Λ , $\mathcal{J}$ and $\mathcal{V}$ as in the proof of Proposition 2. Let $T_{a_\lambda} = Y_{a_\lambda} - p_{a_\lambda}(\sigma - \text{cl } V_\lambda)$ for each $\lambda \in \Lambda$. Then T_{a_λ} is a closed subset of Y_{a_λ} because p_{a_λ} is an open map. Since Y_{a_λ} is normal, there are open sets $M_{a_\lambda, i}$ in Y_{a_λ} such that $T_{a_\lambda} \subset \bigcup_{i=1}^2 M_{a_\lambda, i}$ and $\text{cl } M_{a_\lambda, i} \subset U_{a_\lambda, i}$ for each $i = 1, 2$. Let $C_\lambda = \text{int } p_{a_\lambda}^{-1}(T_{a_\lambda})$. Then

(*) $\{C_\lambda \mid \lambda \in \Lambda\}$ is an open cover of σ .

Proof of ().* This proof is quite similar to that of [1, Theorem 3.4]. For completeness we shall mention this. Let $x = (x_\alpha)_{\alpha < \kappa} \in \sigma$. Then $x \in V_\lambda$ for some λ . Since V_λ is open in σ , there are an $a \in \mathcal{A}$ and an open set W in Y_a such that $x \in p_a^{-1}(W) \subset V_\lambda$. Let $\mu = \lambda \cup \{a\}$. Then $x \in C_\mu$ because $p_a^{-1}(W) \subset p_{a_\mu}^{-1}(T_{a_\mu})$. To show that $p_a^{-1}(W) \subset p_{a_\mu}^{-1}(T_{a_\mu})$, let $y \in p_a^{-1}(W) - p_{a_\mu}^{-1}(T_{a_\mu})$. Then $p_a(y) \in W$ and $p_{a_\mu}(y) \notin T_{a_\mu}$ and so $p_{a_\mu}(y) \in p_{a_\mu}(\sigma - \text{cl } V_\lambda)$. Thus $p_{a_\mu}(y) = p_{a_\mu}(z)$ for some $z \in \sigma - \text{cl } V_\lambda$. Since $a \subset a_\mu$, $p_a(y) = p_a(z)$. Therefore, $p_a(z) \in W$ and so $z \in p_a^{-1}(W)$. Hence $p_{a_\mu}(z) \in T_{a_\mu}$. This is a contradiction. The proof of (*) is complete.

Since $|\Lambda| = \kappa$ and σ is κ -paracompact, there is a locally finite open cover $\{K_\lambda \mid \lambda \in \Lambda\}$ of σ such that $K_\lambda \subset C_\lambda$ for each $\lambda \in \Lambda$. Let

$$L_i = \bigcup_{\lambda \in \Lambda} (K_\lambda \cap p_{a_\lambda}^{-1}(M_{a_\lambda, i})) \quad \text{for } i = 1, 2.$$

Then $\sigma = \bigcup_{i=1}^2 L_i$ and $\text{cl } L_i \subset G_i$. Thus σ is normal.

Proof of Theorem 3. The “if part” follows from (B) in §1. The “only if part” follows from Proposition 3.

Theorem 3 can be generalized as:

Theorem 5. *Suppose every finite subproduct of σ is κ -paracompact and normal. Then σ is κ -paracompact iff σ is normal.*

Proposition 4. *If every finite subproduct of σ is hereditarily metacompact (resp. hereditarily countably metacompact), then σ is hereditarily metacompact (resp. hereditarily countably metacompact).*

Proof. For $n \in \omega$, let $\sigma_n = \{x \in X \mid |\{\alpha \mid x(\alpha) \neq x^*(\alpha)\}| \leq n\}$; σ_n is closed in σ . Let G be an arbitrary open subset of σ and $\mathcal{G}$ an arbitrary (countable) open cover of G . For each $n < \omega$ and $a \in \kappa^n$, choose a point finite open cover $\mathcal{J}_a$ of $(Y_a - \sigma_{n-1}) \cap G$ which is a refinement of $\mathcal{G}$. For each $J \in \mathcal{J}_a$,

choose $G_J \in \mathcal{G}$ such that $J \subset G_J$, and put $\mathcal{H}_n = \{p_a^{-1}(J) \cap G_J - \sigma_{n-1} \mid J \in \mathcal{J}_a, a \in \kappa^n\}$. Then $\mathcal{H}_n$ is a point finite open family in G . To show this, let $x \in G$ and suppose $x \in p_a^{-1}(J) - \sigma_{n-1}$ with $a \in \kappa^n$ and $J \in \mathcal{J}_a$. Then $p_a(x) \in J$ and so $p_a(x) \notin \sigma_{n-1}$. Thus $a \in \{\alpha \in \kappa \mid x(\alpha) \neq x^*(\alpha)\}$.

It is obvious that $\mathcal{H}_n$ is a cover of $(\sigma_n - \sigma_{n-1}) \cap G$. Let $\mathcal{H} = \bigcup_{n < \omega} \mathcal{H}_n$. For each $x \in G$, $x \in \sigma_n - \sigma_{n-1}$ for some n . Thus $x \notin H$ for each $H \in \bigcup_{m \geq n} \mathcal{H}_m$. Therefore, $\mathcal{H}$ is a point finite open cover of G which is a refinement of $\mathcal{G}$.

Proof of Theorem 4. The "if part" follows from Proposition 4 because countable paracompactness coincides with countable metacompactness for normal spaces. The "only if part" follows from Proposition 2.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, SHIZUOKA UNIVERSITY, OHYA,
SHIZUOKA, 422 JAPAN