AVERAGES ALONG UNIFORMLY DISTRIBUTED DIRECTIONS ON A CURVE

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ABSTRACT. We obtain a sharp L^2 estimate for the maximal operator associated with uniformly distributed directions on a curve of finite type in $\mathbf{R}^n$.

INTRODUCTION

Let $\gamma: [0, 1] \rightarrow S^{n-1}$ be a smooth curve crossing each hyperplane of $\mathbf{R}^n$ a finite number of times. If $\mathcal{B}_N$ denotes the family of all cylinders in $\mathbf{R}^n$ having eccentricity N and direction in γ , it is proved in [C] that the maximal operator

$$M_N f(x) = \sup_{x \in R \in \mathcal{B}_N} \frac{1}{|R|} \int_R |f(y)| dy$$

satisfies the estimate

$$(1) \quad \|M_N f\|_{L^2} \leq C_\gamma (\log N)^2 \|f\|_{L^2}$$

where C_γ is independent of N .

The purpose of this note is to show that by imposing an additional condition on γ one can prove a stronger result.

Let γ be a smooth curve satisfying

$$(*) \quad \text{For all } t \in [0, 1], \text{ the set } \{\gamma^{(j)}(t)\}_{0 \leq j < \infty} \text{ spans } \mathbf{R}^n.$$

For a positive integer m let $\mathcal{B}_m$ denote the family of all cylinders in $\mathbf{R}^n$ pointing in the direction of $\gamma(j/2^m)$ for some $0 \leq j \leq 2^m$. let $\mathcal{M}_m f(x) = \sup_{x \in R \in \mathcal{B}_m} (1/|R|) \int_R |f(y)| dy$. Then we will prove the following:

Theorem. *If γ satisfies $(*)$ then*

$$(2) \quad \|\mathcal{M}_m f\|_{L^2} \leq C_\gamma m \|f\|_{L^2}$$

where C_γ is independent of m .

If $n = 2$ or if γ is contained in a 2-dimensional subspace, (2) is known to be true (see [S] or [B]). Also, since $\mathcal{M}_m$ dominates M_{2^m} , (2) implies an improved version of (1).

In what follows all the constants are independent of m .

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AUXILIARY LEMMAS

We will now prove some consequences of (*) that will be used to prove the theorem.

A simple compactness argument shows that if γ satisfies (*), then there exist an integer L and $c > 0$ such that for all $\xi \in S^{n-1}$ and $t \in [0, 1]$

$$(3) \quad \sum_{i=0}^L |\xi \cdot \gamma^{(i)}(t)| \geq c.$$

For $j = 0, 1, 2$ let $\mathcal{U}_j = \{(\xi, t) \in S^{n-1} \times [0, 1] : |\xi \cdot \gamma^{(l)}(t)| \leq c2^{-(l+1)} \text{ for } l \leq j\}$. Then we have

Lemma 1. *There exist $\delta_j > 0$ and $c_j > 0$ such that for all $(\xi, t) \in \mathcal{U}_j$*

$$(4) \quad |s - t| < \delta_j \Rightarrow |\xi \cdot (\gamma^{(j)}(s) - \gamma^{(j)}(t))| \geq c_j |s - t|^{L-j}.$$

Proof. If the lemma is false, we can find sequences $\varepsilon_k \rightarrow 0$, $\delta_k \rightarrow 0$, $(\xi_k, t_k) \in S^{n-1} \times [0, 1]$, and s_k such that $|s_k - t_k| < \delta_k$ and

$$(5) \quad |\xi_k \cdot (\gamma^{(j)}(s_k) - \gamma^{(j)}(t_k))| < \varepsilon_k |s_k - t_k|^{L-j}.$$

Since $\mathcal{U}_j$ is compact, by passing to a subsequence, we can assume that (ξ_k, t_k) converges to $(\xi, t) \in \mathcal{U}_j$. By Taylor's theorem (5) implies that $\xi \cdot \gamma^{(l)}(t) = 0$ for $l = j + 1, \dots, L$. This contradicts (3).

Lemma 1 implies that there exist integers $N_j (\sim \delta_j^{-1})$ such that for all ξ in S^{n-1} the function $\xi \cdot \gamma^{(j)}(t)$ has at most N_j zeros on $\{t \in [0, 1] : |\xi \cdot \gamma^{(l)}(t)| < c2^{-(l+1)} \text{ for } 0 \leq l \leq j-1\}$.

For $\xi \in S^{n-1}$ let $v_\xi(t) = \xi \cdot \gamma(t)$, $\mathcal{Z}_\xi^1 = \{t \in [0, 1] : |v_\xi(t)| > c/2\}$, and $\mathcal{Z}_\xi^2 = \{t \in [0, 1] : |v_\xi(t)| < c/2 \text{ and } |v'_\xi(t)| > c/4\}$. Since $\mathcal{Z}_\xi^1$ and $\mathcal{Z}_\xi^2$ are open (in $[0, 1]$) and disjoint, we can write each $\mathcal{Z}_\xi^j$ as a countable union of disjoint intervals. Since between each two intervals of $\mathcal{Z}_\xi^1$ there exists a t for which either $v_\xi(t) = 0$ or $v'_\xi(t) = 0$, Lemma 1 implies that $\mathcal{Z}_\xi^1$ is the union of at most $N_0 + N_1$ (independent of ξ) intervals. A similar argument applied to $\mathcal{Z}_\xi^2$ in the complement of $\mathcal{Z}_\xi^1$ together with the fact that, on the complement of $\mathcal{Z}_\xi^1 \cup \mathcal{Z}_\xi^2$, $v''_\xi(t)$ has at most N_2 zeros shows that the complement of $\mathcal{Z}_\xi^1 \cup \mathcal{Z}_\xi^2$ can be written as a union of no more than $2(N_0 + N_1 + N_2)$ closed intervals where, on each of these, $v'_\xi(t)$ is monotonic. Let $I = [a, b]$ be one such interval, and let $t_0 \in [a, b]$ be such that $|v'_\xi(t_0)| = \min_I |v'_\xi(t)|$. Then we have

$$\begin{aligned} v'_\xi(t) &= \sum_{j=0}^{L-1} \frac{v_\xi^{(j+1)}(t_0)}{j!} (t - t_0)^j + R_{t_0}(t) \\ &= p'_\xi(t) + R_{t_0}(t) \quad \text{where } |R_{t_0}(t)| \leq C|t - t_0|^L. \end{aligned}$$

Thus if $\delta_y = 1/2 \min\{\min_j \delta_j, c_1 C^{-1}\}$ we have for $|t - t_0| < \delta_y$ and $t \neq t_0$

$$\left| \frac{p'_\xi(t)}{v'_\xi(t)} - 1 \right| \leq \frac{C|t - t_0|^L}{|v'_\xi(t)|} \leq \frac{1}{2},$$

which implies

$$|p'_\xi(t)| \leq |v'_\xi(t)| \leq 2|p'_\xi(t)|.$$

If we let $p'_\xi(t) - v'_\xi(t_0) = q'_\xi(t)$, we have by Lemma 1 that there exist $c_\xi > 0$ such that $|q'_\xi(t)| \approx c_\xi |t - t_0|^k$ for $|t - t_0| < \delta_\gamma$ and for some k with $1 \leq k \leq L - 1$. If $|t - t_0| > \delta_\gamma$ and $t \in I$, Lemma 1 implies that $|v'_\xi(t)| \geq c_1 \delta_\gamma^{L-1}$. Since $v_\xi(t)$ has at most two zeros on I , we can divide $\{t \in I: |t - t_0| < \delta_\gamma\}$ in no more than four intervals where $v_\xi(t)$ is monotonic and of constant sign satisfying estimates like the above. Thus, if we let $N_\gamma = 10(N_0 + N_1 + N_2)$ and $c_\gamma = \min\{c/4, c_1 \delta_\gamma^{L-1}\}$, we obtain

Lemma 2. *There exist an integer N_γ and $c_\gamma > 0$ such that for all ξ in S^{n-1} we have*

$$[0, 1] = U_\xi^1 \cup \dots \cup U_\xi^{N_\xi} \cup V_\xi^1 \cup \dots \cup V_\xi^{M_\xi} \cup W_\xi^1 \cup \dots \cup W_\xi^{K_\xi}$$

where $N_\xi + M_\xi + K_\xi \leq N_\gamma$ and where the U_ξ^i 's, V_ξ^i 's, and W_ξ^i 's are closed intervals with disjoint interiors for which

- (i) $|v_\xi(t)| \geq c_\gamma$ on $\bigcup_i U_\xi^i$,
- (ii) $|v'_\xi(t)| \geq c_\gamma$ on $\bigcup_i V_\xi^i$, and
- (iii) for each $i \leq K_\xi$ there exist $c_\xi^i > 0$, $t_0 \in W_\xi^i$, and $k = k_{\xi, i, t_0}$ with $1 \leq k \leq L$ such that

$$|v_\xi(t)| \approx |v_\xi(t_0)| + c_\xi^i |t - t_0|^k \quad \text{and} \quad |v'_\xi(t)| \approx c_\xi^i k |t - t_0|^{k-1}.$$

Proof of Theorem. The proof is based in a square function argument following the ideas in [W, NSW].

Let $\varphi \in C_0^\infty(\mathbf{R})$ be nonnegative, with $\varphi \equiv 1$ on $[-\frac{1}{4}, \frac{1}{4}]$ and such that $\int_{-\infty}^\infty \varphi(t) dt = 1$. For $h > 0$ let $\varphi_h(t) = h^{-1} \varphi(h^{-1}t)$ and let $\omega_j = \gamma(j2^{-m})$.

For $0 \leq j \leq 2^m$ let

$$T_{h,j}^m f(x) = \int_{-\infty}^\infty f(x - t\omega_j) \varphi_h(t) dt, \quad T^m f(x) = \sup_{h,j} |T_{h,j}^m f(x)|.$$

Then a simple geometric argument shows that it suffices to prove that

$$(6) \quad \|T^m f\|_{L^2} \leq C_\gamma m \|f\|_{L^2} \quad \text{for } f \geq 0.$$

For $m = 1$, (6) follows from the boundedness of the one-dimensional Hardy-Littlewood maximal operator. Suppose (6) is true for $m = 1$. Then for $f \geq 0$

$$(7) \quad \begin{aligned} T^m f(x) &\leq T^{m-1} f(x) + \sup_{h,j} |T_{h,2j}^m f(x) - T_{h,2j-1}^m f(x)| \\ &= T^m f(x) + \sup_{h,j} H_{h,j}^m f(x) \end{aligned}$$

and (6) will follow if we can show that

$$(8) \quad \left\| \sup_{h,j} H_{h,j}^m f \right\|_{L^2} \leq C_\gamma \|f\|_{L^2}.$$

For $j = 1, \dots, 2^m$, let Γ_j be the cone $\{\xi \in \mathbf{R}^n: |\xi \cdot \omega_j| > c_1 2^{-mL} |\xi|\}$ and let K_j be the complement of Γ_j , and for $j = 1, \dots, 2^{m-1}$, let $f(x) = f_j(x) + r_j(x)$ where $\hat{r}_j(\xi) = \chi_{K_{2j} \cup K_{2j-1}}(\xi) \hat{f}(\xi)$. $\hat{\cdot}$ denotes the Fourier transform.

An argument similar to the one in [W, p. 88] shows that

$$\sup_{h,j} H_{h,j}^m f(x) \leq C(g_1(f)(x) + g_2(f)(x))$$

where

$$g_1(f)(x) = \left(\sum_{j=1}^{2^{m-1}} \left(\sup_h T_{h,2j}^m |r_j(x)| + \sup_h T_{h,2j-1}^m |r_j(x)| \right)^2 \right)^{1/2},$$

$$g_2(f)(x) = \left(\int_0^\infty \sum_{j=1}^{2^{m-1}} |T_{h,2j}^m f_j(x) - T_{h,2j-1}^m f_j(x)|^2 \frac{dh}{h} \right)^{1/2}.$$

Thus (8) and hence the theorem will be a consequence of the following two estimates:

$$(9) \quad \|g_1(f)\|_{L^2} \leq C_\gamma \|f\|_{L^2},$$

$$(10) \quad \|g_2(f)\|_{L^2} \leq C_\gamma \|f\|_{L^2}.$$

Proof of (9). By the boundedness of the one-dimensional Hardy-Littlewood maximal operator and Plancherel's theorem, one has

$$(11) \quad \|g_1(f)\|_{L^2}^2 \leq C \int_{\mathbf{R}^n} \sum_{j=1}^{2^{m-1}} |r_j(x)|^2 dx = C \int_{\mathbf{R}^n} \sum_{j=1}^{2^{m-1}} \chi_{K_{2j} \cup K_{2j-1}}(\xi) |\hat{f}(\xi)|^2 d\xi.$$

Since the K_j 's are conic, it is enough to prove that no $\xi \in S^{n-1}$ belongs to more than C_γ of the K_j 's. Given ξ in K_{j_0} we have $|v_\xi(t)| \leq c_1 2^{-mL}$. By (4), if $k > c_1^{1/L}$, then ξ does not belong to $K_{j_0 \pm k}$. Thus ξ does not belong to more than $N_\gamma c_1^{1/L}$ of the K_j 's.

Proof of (10). Plancherel's theorem implies that

$$(12) \quad \|g_2(f)\|_{L^2}^2 = \int_{\mathbf{R}^n} \sum_{j=1}^{2^{m-1}} \int_0^\infty |\hat{\phi}(h\xi \cdot \omega_{2j}) - \hat{\phi}(h\xi \cdot \omega_{2j-1})|^2 |\hat{f}_j(\xi)|^2 \frac{dh}{h} d\xi$$

$$= \int_{\mathbf{R}^n} m(\xi) |\hat{f}(\xi)|^2 d\xi,$$

where

$$(13) \quad m(\xi) = \sum_{j=1}^{2^{m-1}} \int_0^\infty |\hat{\phi}(h\xi \cdot \omega_{2j}) - \hat{\phi}(h\xi \cdot \omega_{2j-1})|^2 \chi_{\Gamma_{2j} \cap \Gamma_{2j-1}}(\xi) \frac{dh}{h},$$

and we are left to prove that $m(\xi) \leq C_\gamma$. This is accomplished by dividing the curve γ in pieces where one has control over the decay of $\hat{\phi}(h\xi \cdot \omega_j)$ in ξ and j in estimating (13). The details are below.

By Lemma 2 we can, for each ξ , split the sum in (13) in no more than N_γ sums of the form $\sum_{j \in 2^{-m} U_\xi^i}$, $\sum_{j \in 2^{-m} V_\xi^i}$, and $\sum_{j \in 2^{-m} W_\xi^i}$. Thus the theorem will follow if we can show that each of these sums is bounded with bound independent of m and ξ . By homogeneity we only need to consider $\xi \in S^{n-1}$.

Since $|\xi \cdot \omega_j| \geq c_\gamma$ for $j2^{-m} \in U_\xi^i$, and since $\hat{\phi}$ is a Schwartz function, we have that $|\hat{\phi}(h\xi \cdot \omega_{j+1}) - \hat{\phi}(h\xi \cdot \omega_j)|^2 \leq Ch^2|\omega_{j+1} - \omega_j|^2|\hat{\phi}'(h\xi \cdot u_j)|^2$ with $|\xi \cdot u_j| \geq c_\gamma$. This implies

$$(14) \quad \sum_{j2^{-m} \in U_\xi^i} \int_0^\infty |\hat{\phi}(h\xi \cdot \omega_{j+1}) - \hat{\phi}(h\xi \cdot \omega_j)|^2 \frac{dh}{h} \leq C_\gamma.$$

We now prove a similar estimate for W_ξ^i . There is no lack of generality in assuming that $W_\xi^i = [0, \varepsilon]$ and that $v_\xi(0) = 0$. Lemma 2 implies that for $j2^{-m} \in W_\xi^i$

$$(15) \quad |\xi \cdot (\omega_{2j} - \omega_{2j-1})| \leq Cc_\xi j^{k-1} 2^{-mk},$$

$$(16) \quad |\xi \cdot \omega_j| \geq Cc_\xi j^k 2^{-mk}.$$

Since $\hat{\phi}$ is smooth and rapidly decreasing, by (15) and (16) we obtain

$$(17) \quad |\hat{\phi}(h\xi \cdot \omega_{2j}) - \hat{\phi}(h\xi \cdot \omega_{2j-1})|^2 \leq Cc_\xi^2 j^{2(k-1)} 2^{-2mk} h^2,$$

$$(18) \quad |\hat{\phi}(h\xi \cdot \omega_j)|^2 \leq C_\alpha c_\xi^{-2\alpha} j^{-2k\alpha} 2^{-2mk\alpha} h^{-2\alpha}.$$

Splitting each integral in $\int_0^{a_j} + \int_{a_j}^\infty$ where the a_j 's are to be determined later and using (17) and (18) on each integral respectively we obtain that $\sum_{j2^{-m} \in W_\xi^i}$ is dominated by

$$(19) \quad \sum_{j2^{-m} \in W_\xi^i} Cc_\xi^2 j^{2(k-1)} 2^{-2mk} a_j^2 + C_\alpha c_\xi^{-2\alpha} j^{-2k\alpha} 2^{-2mk\alpha} a_j^{-2\alpha}.$$

To finish, put $\beta = k - \frac{1}{4}$, $\alpha = 3$, and let $a_j = c_\xi^{-1} j^{-\beta} 2^{mk}$ in (19) obtaining

$$(20) \quad \sum_{j2^{-m} \in W_\xi^i} \leq C \sum_1^{2^{m+1}} j^{2(k-\beta-1)} + j^{2\alpha(\beta-k)} \leq C \sum_1^\infty j^{-3/2}.$$

The terms $\sum_{j2^{-m} \in V_\xi^i}$ can be handled similarly with $k = 1$.

Since (20) is independent of ξ and m , the proof is complete.

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