GENERALIZED HADWIGER NUMBERS FOR SYMMETRIC OVALS

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ABSTRACT. Some estimations for the “juxtaposition function” h_F and an asymptotic formula for the function h_F/h_G , where F, G are central symmetric convex bodies, are given. Hadwiger and Grünbaum gave for $h_F(1)$ the bounds $n^2 + n \leq h_F(1) \leq 3^n - 1$. Grünbaum conjectured (and proved for $n = 2$ in Pacific J. Math. **11** (1961), 215–219) that for every even r between these bounds there exists in E^n an oval F such that $h_F(1) = r$. Lower bounds for h_F could be derived in the same way as in Theorems 1 and 2 from a good estimate of packing numbers on a Minkowski sphere, that is, from solutions to a Tammes-type problem in a Banach space.

For a topological disk $F \subseteq \mathbb{E}^n$ we shall denote by $h_F: (0, 1] \rightarrow \mathbb{N}$ the “juxtaposition function” introduced by the first author [2, 3] as follows. Let $A_{F,\lambda}$ denote the family of all sets, homothetic to F in the ratio λ , which have only boundary points in common with F . Then $h_F(\lambda)$ is the greatest integer k such that $A_{F,\lambda}$ contains k sets with pairwise disjoint interiors. In particular, $h_F(1)$ is just the Hadwiger number of F .

In case of convex F , Hadwiger [11] and Grünbaum [8] gave for $h_F(1)$ the bounds $n^2 + n \leq h_F(1) \leq 3^n - 1$. Grünbaum [8] conjectured (and proved for $n = 2$; see also Boltjanski and Gohberg [4]) that for every even r between these bounds there exists in $\mathbb{E}^n$ an oval F such that $h_F(1) = r$.

Unless explicitly stated otherwise, throughout this paper F, G will denote symmetric plane ovals. Any such F determines a norm $\|\cdot\|_F$ by $\|x - y\|_F = \|x - y\|/\|o - z\|$, where $\|\cdot\|$ is the Euclidean norm, o is the center of F , and z is a point on the boundary ∂F of F such that oz and xy are parallel. With this norm $\mathbb{E}^2$ becomes a Banach space, with unit disk isometric to F . Set $p(F)$ for the perimeter of ∂F in its inner norm.

Theorem 1. *For a symmetric oval F in the plane*

$$(1) \quad p(F) = 2 \lim_{\lambda \rightarrow 0} \lambda h_F(\lambda).$$

Proof. Let x, y be points of ∂F , and let points x', y' be given by $ox' = (1 + \lambda)ox$ and $oy' = (1 + \lambda)oy$. Denote by F_x, F_y those sets in $A_{F,\lambda}$ which have centers at x' and y' , respectively. If $F_x \cap F_y \neq \emptyset$, it follows from the

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symmetry and convexity of F that $x'y' \subset F_x \cap F_y$. We put $x'y' \cap \partial F_x = \{a, b\}$, $x'y' \cap \partial F_y = \{c, d\}$, and $z \in \partial F$ such that oz is parallel to $x'y'$. Then

$$\|x' - y'\| \leq \|x' - b\| + \|c - y'\| = 2\|x' - b\| = 2\lambda\|o - z\|,$$

hence

$$\|x - y\|_F = \|x' - y'\|/(1 + \lambda)\|o - z\| \leq 2\lambda/(1 + \lambda).$$

Reversing the reasoning we obtain

$$(2) \quad \text{int } F_x \cap \text{int } F_y = \emptyset \quad \text{if and only if} \quad \|x - y\|_F \leq 2\lambda/(1 + \lambda).$$

Now consider a maximal collection $\{F_i: i = 1, \dots, k\} \subset A_{F, \lambda}$ of sets with disjoint interiors and the points $x_i \in \partial F$, $i = 1, \dots, k$, for which $F_i = F_{x_i}$. From (2) it follows that $\|x_i - x_{i+1}\|_F \leq 2\lambda/(1 + \lambda)$ and thus

$$\sum_{1 \leq i \leq k} \|x_i - x_{i+1}\|_F \leq 2k\lambda/(1 + \lambda);$$

however,

$$p(k, F) = \sup \left\{ \sum_{1 \leq i \leq k} \|x_i - x_{i+1}\|_F, \quad x_i \in \partial F \right\} \leq p(F).$$

These inequalities yield

$$(3) \quad h_F(\lambda) = k \leq (1 + \lambda)p(k, F)/2\lambda < (1 + \lambda)p(F)/2\lambda.$$

Conversely, let P_λ be an inscribed polygon with $2k$ vertices $u_1, \dots, u_{2k}$ such that P_λ is symmetric about o and

$$\begin{aligned} \|u_1 - u_2\|_F &= \|u_2 - u_3\|_F = \dots = \|u_{k-2} - u_{k-1}\|_F = 2\lambda/(1 + \lambda), \\ 2\lambda/(1 + \lambda) &\leq \|u_{k-1} - u_k\|_F < 4\lambda/(1 + \lambda). \end{aligned}$$

Then the sets F_{u_i} have disjoint interiors and

$$4(k + 1)\lambda/(1 + \lambda) > \sum_{1 \leq i \leq 2k} \|u_i - u_{i+1}\|_F \geq 4k\lambda/(1 + \lambda).$$

Since $h_F(\lambda) \geq 2k$, it follows that

$$(4) \quad 2 + h_F(\lambda) \geq (1 + \lambda) \left(\sum_{1 \leq i \leq 2k} \|u_i - u_{i+1}\|_F \right) / 2\lambda.$$

If $p(\lambda)_F$ denotes the perimeter of P_λ in the $\|\cdot\|_F$ norm, then (see [1, 11])

$$(5) \quad \lim_{\lambda \rightarrow 0} p(\lambda)_F = p(F).$$

For symmetric ovals F, G relations (3)–(5) imply

$$(6) \quad \lim_{\lambda \rightarrow 0} h_F(\lambda)/h_G(\lambda) \geq \lim_{\lambda \rightarrow 0} ((-2 + (1 + \lambda)p(\lambda)_F)/2\lambda)/((1 + \lambda)p(G))/2\lambda \\ = p(F)/p(G),$$

and similarly the reverse inequality. Therefore, taking for G a square we obtain the claim which was to be proved.

Denote by $[t]$ the integer part of $t \in \mathbb{R}$.

Theorem 2. *For every symmetric oval F in the plane*

$$(7) \quad 3 + [3/\lambda] \leq h_F(\lambda) \leq 4(1 + \lambda)/\lambda,$$

with equality on the left if and only if $1/\lambda \in \mathbb{N}$, and F is an affine-regular hexagon and equality on the right if and only if $1/\lambda \in \mathbb{N}$ and F is a parallelogram.

Proof. A result of Golab [6] and Reshetnyak [14], generalized by Schäffer [15], asserts that $6 \leq p(F) \leq 8$. Hence we have

$$h_F(\lambda) \leq 4(1 + \lambda)/\lambda,$$

and, using the existence of an affine-regular hexagon inscribed in F [13], we obtain $h_F(2/(1 + k)) \geq 6k$. Since $h_F(\lambda)$ is a decreasing function of λ , we are done.

If the dimension of F is greater than two, the situation is essentially different. We shall prove (see also [7])

Theorem 3. *Any symmetric convex body $F \subset \mathbb{E}^n$ satisfies the inequality*

$$(8) \quad h_F(\lambda) \leq ((1 + \lambda)^n - 1)/\lambda^n,$$

with equality if and only if $1/\lambda \in \mathbb{N}$ and F is a parallelotical body.

Proof. Let $B_\lambda = \bigcup_{H \in A_{F,\lambda}} H$. We shall prove that

$$(9) \quad B_\lambda \subset (1 + 2\lambda)F.$$

Indeed, let x be a point on the boundary of F_v , $|ox| \cap \partial F = \{a\}$, $|ov| \cap \partial F_v = \{q\}$, and let vx'' be parallel to qx with $x'' \in \partial F$. Then $\angle qv'x = \angle v'xo + \angle v'ox \geq \angle v'ox$, which yields $\angle vox \leq \angle vox'' = \angle qv'x$. Since F is convex, we can take a point b in the nonempty intersection $|oa| \cap |vx''|$. Then $|vx''| \subset F$, $b \in F$, $b \in |ox|$. Since

$$\|o + a\|/\|o - x\| \geq \|o - b\|/\|o - x\| \geq \|o - v\|/\|o - q\| = 1/(1 + 2\lambda),$$

the point x belongs to $(1 + 2\lambda)F$, and (9) is proved.

If $\{F_i, i = 1, \dots, k\} \subset A_{F,\lambda}$ have disjoint interiors, then

$$\bigcup_{1 \leq i \leq k} F_i \subset B_\lambda \subset (1 + 2\lambda)F;$$

therefore,

$$\text{vol}(F) + \text{vol}(F_1) + \dots + \text{vol}(F_k) \leq (1 + 2\lambda)^n \text{vol}(F)$$

where $\text{vol}(F)$ denotes the volume of F . This gives the desired estimation on $h_F(\lambda)$. The equality case is treated in [7].

Lower bounds for $h_F(\lambda)$ could be derived in the same way as in Theorems 1 and 2 from a good estimate of packing numbers on a Minkowski sphere, that is, from solutions to a Tammes-type problem in a Banach space.

Grünbaum asked what happens to relation (1) in case F is not centrally symmetric. We recall that for an arbitrary oval F and $z \in \text{int } F$ a norm (nonsymmetric, in general) is defined by the Minkowski functional

$$\|x\|_{F,z} = \inf\{\lambda > 0 : x - z \in \lambda(F - z)\}.$$

Using the (possibly nonsymmetric) distance derived from this norm it is possible to define arc-length for oriented arcs. For an oriented closed curve C let the length of C in the metric derived from $\|\cdot\|_{F,z}$ be denoted by $p_{F,z}(C)$. The intrinsic perimeter (self-circumference [6, 10]) of F is $P(F) = \inf\{p_{F,z}(\partial F) : z \in \text{int } F\}$. Then it follows that

$$g(F) = \lim_{\lambda \rightarrow 0} \lambda h_F(\lambda) / P(F)$$

is a measure of symmetry (see [8]). By the same method as used above, it is possible to show that $g(F) \leq \frac{1}{2}$, with equality if F is centrally symmetric. If F is a triangle then $g(F) = \frac{1}{3}$, and we conjecture that $g(F) \geq \frac{1}{3}$ for any oval F .

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