A CLASS OF RIESZ SETS

DANIEL LI

(Communicated by Dale E. Alspach)

ABSTRACT. Let G be a metrizable compact abelian group. A subset Λ in the dual group is said to be ergodic if every $f \in L^\infty(G)$ whose spectrum lies in a translate of Λ has a unique invariant mean. It is shown that such a set is a Riesz set.

INTRODUCTION

It is well known that Rosenthal sets are Riesz sets [1, 4, 3]. A natural class of sets is lying between them; it is connected with the possibility for an element of $L^\infty(G)$ to have a unique invariant mean. A subset Λ of a discrete abelian group is said to be ergodic if every $f \in L^\infty(G)$ whose spectrum lies in a translate of Λ has a unique invariant mean. Katznelson [8] proved that $\mathbb{N}$, which is a Riesz set, is not an ergodic set. A few years ago, Françoise Lust-Piquard [6] showed that $\mathbb{P} \cap (5\mathbb{Z} + 3)$, $\mathbb{P}$ being the set of primes, is an ergodic set which is not a Rosenthal set. Of course, each Rosenthal set is ergodic since continuous functions have a unique invariant mean. In view of these results, the interplay between ergodic and Riesz sets arises naturally. We prove in this paper that every ergodic set is a Riesz set.

DEFINITIONS AND NOTATION

Throughout the paper G will be a metrizable compact abelian group and $\Gamma = \widehat{G}$ will be its countable discrete dual group.

As usual $\mathcal{M}_\Lambda$, L_Λ^1 , $\mathcal{E}_\Lambda$, L_Λ^∞ are the subspaces consisting of the elements of $\mathcal{M}(G)$, $L^1(G)$, $\mathcal{E}(G)$, $L^\infty(G)$ respectively whose spectrum (i.e., the support of their Fourier transform) lies in $\Lambda \subseteq \Gamma$.

A subset $\Lambda \subseteq \Gamma$ is said to be a *Riesz set* if each measure $\mu \in \mathcal{M}_\Lambda$ is absolutely continuous (w.r.t. the Haar measure); in short, $\mathcal{M}_\Lambda = L_\Lambda^1$; it is said to be a *Rosenthal set* if each $f \in L_\Lambda^\infty$ is represented by a continuous function; in short, $L_\Lambda^\infty = \mathcal{E}_\Lambda$.

An *invariant mean* on L^∞ is a translation-invariant continuous linear form on L^∞ such that $\|M\| = M(1) = 1$. An element $f \in L^\infty$ is said to be *ergodic* if all the invariant means take the same value on it; it is said to be *totally ergodic*

Received by the editors March 17, 1992.

1991 *Mathematics Subject Classification*. Primary 43A46; Secondary 28C10, 46B20, 04A15.

if γf is ergodic for every $\gamma \in \Gamma$. For such an f , we have $M(\gamma f) = \hat{f}(-\gamma)$ for each invariant mean M .

Definition. $\Lambda \subseteq \Gamma$ is said to be an *ergodic set* if every $f \in L^\infty_\Lambda$ is totally ergodic.

ERGODIC AND RIESZ SETS

The main result is

Theorem 1. *Let G be a metrizable compact abelian group. Every ergodic set in $\Gamma = \widehat{G}$ is a Riesz set.*

Proof. For every $\mu \in \mathcal{M}_\Lambda$, we write $\mu = \mu_a + \mu_s$, its Lebesgue-Radon-Nikodým decomposition. For every $f \in L^\infty$, we have $\mu * f \in L^\infty_\Lambda$ and so $\mu * f$ is totally ergodic. Since $\mu_a * f$ is totally ergodic, as a continuous function, we obtain that $\mu_s * f$ is totally ergodic. The following proposition allows us to conclude that $\mu_s = 0$. $\square$

Proposition 2. *Let G be a metrizable compact abelian group. For every singular measure $\sigma \neq 0$, there is an $f \in L^\infty$ such that $\sigma * f$ is not totally ergodic.*

Proof. First, let μ be a singular measure such that $\theta = \mu(G) \neq 0$.

Since the measure $\nu = |\mu|$ is singular, there is a Borel set B such that $m(B) = 0$ (m being the Haar measure on G) and such that $\nu(B) = \|\nu\|$. It will be convenient to take B symmetric.

Let D be a countable dense subgroup of G . We set $A = B + D$; since D is countable, we have $m(A) = 0$, and since $B \subseteq A$, we have $\nu(A) = \|\nu\|$.

Let now Ω be an open set of G containing A and such that $m(\Omega) \leq 1/4$.

We are going to show that $\mu * \mathbf{1}_\Omega$ is not ergodic. For this, we shall use the fact that an element $f \in L^\infty$ is ergodic if and only if there is a constant function in the norm-closed convex hull of the orbit of f by G [5, 8].

Let $\lambda_1, \dots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$ and $x_1, \dots, x_n \in G$ be given. For each $i \in \{1, \dots, n\}$, the function φ_i defined by

$$\varphi_i(x) = (\mathbf{1}_{\Omega+x_i} * \nu)(x) = (\mathbf{1}_\Omega * \nu)_{x_i}(x) = \nu(\Omega + x - x_i)$$

is l.s.c.; therefore the set

$$U_i = \{x \in G ; \nu(\Omega + x - x_i) > \|\nu\| - |\theta|/4\}$$

is open. Moreover, it is dense since it contains $D + x_i$. Then, $U = \bigcap_{i=1}^n U_i$ is also a dense open set, so $m(U) > 0$.

Moreover, for each $x \in U$, we have

$$\forall i \in \{1, \dots, n\} \quad \nu(\Omega + x - x_i) > \|\nu\| - |\theta|/4.$$

We have then, for each $x \in U$,

$$\begin{aligned} \left| \sum_{i=1}^n \lambda_i \mu(\Omega + x - x_i) - \mu(\Omega) \right| &= \left| \sum_{i=1}^n \lambda_i [\mu(\Omega + x - x_i) - \mu(\Omega)] \right| \\ &\leq \sum_{i=1}^n \lambda_i \nu((\Omega + x - x_i) \Delta \Omega) < \frac{|\theta|}{4} \end{aligned}$$

and so, since $\mu(\Omega) = \mu(G) = \theta$,

$$\left| \sum_{i=1}^n \lambda_i \mu(\Omega + x - x_i) \right| > \frac{3|\theta|}{4}.$$

Therefore,

$$\left\| \sum_{i=1}^n \lambda_i (1_\Omega * \mu)_{x_i} \right\|_\infty > \frac{3|\theta|}{4}.$$

Hence, there is no constant function in the norm-closed convex hull of $\{(1_\Omega * \mu)_x; x \in G\}$, since if there is one, namely, $C1$, we do have by the above calculations $C \geq 3|\theta|/4$; but, on the other hand, we also have

$$C = \int_G (1_\Omega * \mu) dm = \mu(G)m(\Omega),$$

and so $|C| \leq |\theta|/4$.

This proves that $1_\Omega * \mu$ is not ergodic.

To finish the proof, let $\gamma \in \Gamma$ be such that $\hat{\sigma}(\gamma) \neq 0$. Setting $\mu = \bar{\gamma}\sigma$, we have $\mu(G) \neq 0$, and then the above proof shows that

$$\bar{\gamma}[\sigma * (\gamma 1_\Omega)] = (\bar{\gamma}\sigma) * 1_\Omega$$

is not ergodic; therefore, $\sigma * (\gamma 1_\Omega)$ is not totally ergodic. $\square$

Corollary 3. *Let Γ be a countable (infinite) discrete abelian group. The set $\mathcal{Erg}(\Gamma)$ of all the ergodic sets in Γ is a coanalytic subset of $\mathcal{P}(\Gamma)$ which is not a Borel subset.*

Proof. From [2, Theorem 2] and Theorem 1 above, $\mathcal{Erg}(\Gamma)$ is not an analytic subset of $\mathcal{P}(\Gamma)$. It is then enough to show that it is coanalytic.

An element $f \in L^\infty(G)$ is ergodic if and only if for every $k \geq 1$ there is an integer n and elements $x_1, \dots, x_n$ in the group G such that

$$(*) \quad \left\| \frac{1}{n} \sum_{i=1}^n f_{x_i} - \hat{f}(0)1 \right\|_\infty \leq \frac{1}{k};$$

hence, the set $E(G)$ of all the totally ergodic elements of $L^\infty(G)$ can be written as

$$E(G) = \bigcap_{\gamma \in \Gamma} \bigcap_{k \geq 1} \bigcup_{n \geq 1} \bigcup_{M \geq 1} F_{k,n,M},$$

where

$$F_{k,n,M} = \{f \in L^\infty(G); \|f\|_\infty \leq M \text{ and } \exists x_1, \dots, x_n \in G: (*)\}.$$

$F_{k,n,M}$ is w^* -compact, as the projection of the compact

$$\{(x_1, \dots, x_n, f) \in G^n \times (B_{L^\infty(G)}(0, M), w^*) : (*)\}.$$

This shows that $E(G)$ is a $K_{\sigma\delta}$. To finish the proof, it suffices to check that $\mathcal{P}(\Gamma) \setminus \mathcal{Erg}(\Gamma)$ is the projection of the $G_{\delta\sigma}$:

$$\{(\Lambda, f) \in \mathcal{P}(\Gamma) \times (L^\infty(G), w^*) : \hat{f} = 0 \text{ on } \Lambda^c \text{ and } f \in L^\infty(G) \setminus E(G)\}$$

and so is analytic. $\square$

Remark. As was recalled in the introduction, every Rosenthal set is a Riesz set. More precisely, we have the following implications:

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v)$$

between the properties:

- (i) Λ is a Rosenthal set;
- (ii) the space $L^1/L^1_{\Gamma \setminus (-\Lambda)}$ contains no isomorphic copy of l_1 (since its dual space L^∞_Λ has the Radon-Nikodým Property);
- (iii) every element of L^∞_Λ is represented by a Riemann-integrable function (by [7, Corollary IV.4]);
- (iv) Λ is an ergodic set (since every Riemann-integrable function is totally ergodic; see, for instance, [8] or [9]);
- (v) Λ is a Riesz set (by Theorem 1).

ACKNOWLEDGMENT

I am grateful to Françoise Lust-Piquard, Gilles Godefroy, and Michel Talagrand for fruitful conversations.

REFERENCES

1. R. E. Dressler and L. Pigno, *Rosenthal sets and Riesz sets*, Duke Math. J. **41** (1974), 675–677.
2. G. Godefroy, *On coanalytic families of sets in harmonic analysis*, Illinois J. Math. **35** (1991), 241–249.
3. F. Lust, *Ensembles de Rosenthal et ensembles de Riesz*, C. R. Acad. Sci. Paris **282** (1976), 833–835.
4. F. Lust-Piquard, *Propriétés géométriques des sous-espaces invariants par translation de $L^1(G)$ et $C(G)$* , Sémin. Géom. Esp. Banach École Polytechn., Mimeog. Ecole Polytechnique, Palaiseau, France, 1977–78, exposé no. 26.
5. ———, *Eléments ergodiques et totalement ergodiques dans $L^\infty(G)$* , Studia Math. **69** (1981), 191–225.
6. ———, *Bohr local properties of $C_\Lambda(\mathbb{T})$* , Colloq. Math. **58** (1989), 29–38.
7. F. Lust-Piquard and W. Schachermayer, *Functions in $L^\infty(G)$ and associated convolutions operators*, Studia Math. **93** (1989), 109–136.
8. L. A. Rubel and A. L. Shields, *Invariant subspaces of L^∞ and H^∞* , J. Reine Angew. Math. **272** (1975), 32–44.
9. M. Talagrand, *Some functions with a unique invariant mean*, Proc. Amer. Math. Soc. **82** (1981), 253–256.

ANALYSE HARMONIQUE, UNIVERSITÉ PARIS-SUD, BAT.425, 91405 ORSAY CEDEX, FRANCE
E-mail address: li@anh.matups.fr

EQUIPE D'ANALYSE, UNIVERSITÉ PARIS VI, TOUR 46-0 4ÈME ÉTAGE, 75252 PARIS CEDEX, FRANCE
E-mail address: li@matups.matups.fr