A NOTE ON COMPACT IDEAL PERTURBATIONS IN SEMIFINITE VON NEUMANN ALGEBRAS

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ABSTRACT. Let M be a semifinite von Neumann algebra and denote by $J(M)$ the closed two-sided ideal generated by the finite projections in M . A subspace $S \subset M$ is called local if it is equal to the ultraweak closure of $S \cap J(M)$. If $M = B(H)$ and $J(M) = K(H)$, Fall, Arveson, and Muhly proved that $S + J(M)$ is closed for every local subspace S .

In this note we prove that if M is a type II_∞ factor, then there exist local subspaces in M which fail to have closed compact ideal perturbations; thus answering in the negative a question of Kaftal, Larson, and Weiss.

1. INTRODUCTION

It is well known that if I is a closed two-sided ideal in a C^* -algebra A and if B is a C^* -subalgebra of A then $B + I$ is closed [3]. The proof relies on the fact that a one-to-one $*$ -homomorphism between two C^* -algebras is automatically isometric.

This is no longer the case if we drop the assumption that B is selfadjoint. However, if $A = B(H)$, the algebra of bounded operators on a Hilbert space H , and if $I = K(H)$, the ideal of compact operators, certain classes of (non-selfadjoint) algebras are known to have closed compact perturbations.

Attention has been especially devoted to the class of reflexive algebras. If $L \subset B(H)$ is a strongly closed lattice of projections containing 0 and I , define

$$\text{Alg } L = \{x \in B(H); (I - p)xp = 0 \text{ for every } p \in L\},$$

and, dually, if A is a subalgebra of $B(H)$, define

$$\text{Lat } A = \{p \in B(H) \text{ projection}; (I - p)xp = 0 \text{ for every } x \in A\}.$$

A subalgebra $A \subset B(H)$ is said to be reflexive if $A = \text{Alg Lat } A$.

It is known that nest algebras (i.e., reflexive algebras with $\text{Lat } A$ totally ordered) have closed compact perturbations [4] and also certain classes of CSL-algebras (i.e., reflexive algebras with $\text{Lat } A$ commutative) (see [6, 9]).

The strongest result in this direction is due to Fall, Arveson, and Muhly [4] and states that if S is equal to the ultraweak closure of $S \cap K(H)$, then $S + K(H)$ is closed. Such subspaces were called local [4]. The proof heavily

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used the dualities $K(H)^* = C^1(H)$ and $C^1(H)^* = B(H)$. ($C^1(H)$ is the ideal of trace-class operators on H .)

Notably, all the examples of CSL-algebras previously mentioned were proved to be local. Several examples of reflexive algebras are known to have nonclosed compact perturbations [1, 5], but all of them were shown to contain no nonzero compact operators. Let us also mention here the further deep results of Davidson and Power on perturbations and M -ideals in C^* -algebras [2].

If, more generally than $B(H)$, we consider a semifinite von Neumann algebra M , define $J(M)$ to be the norm-closed two-sided ideal generated by the finite projections in M . This ideal, called the compact ideal space of M , behaves in many ways like $K(H)$, but there also is notable difference, and completely new techniques are sought in this case (see, e.g., [10]).

If $L \subset M$ is a totally ordered strongly closed family of projections containing 0 and I (i.e., a nest), then $M \cap \text{Alg } L$ is called the nest-subalgebra of M corresponding to L [7].

Define a subspace $S \subset M$ to be local if S equals the ultraweak closure of $S \cap J(M)$.

The first perturbation result in semifinite algebras was recently obtained by Kaftal, Larson, and Weiss [8], who proved that if $A \subset M$ is a nest-subalgebra, then $A + J(M)$ is norm-closed.

The major difficulty of the proof was that duality arguments were no longer available, since $J(M)^{**} \neq M$; although they were able to show that nest-subalgebras of semifinite von Neumann algebras are local. The question whether a Fall-Arveson-Muhly-type theorem holds true arose naturally:

Problem [8]. If M is semifinite and if $S \subset M$ is a local subspace, is $S + J(M)$ closed?

The purpose of this paper is to answer this question in the negative, thus leading to the conclusion that in the continuous semifinite case perturbation problems are more difficult than in the discrete case, and new techniques, which might differ from case to case, are required.

2. A COUNTEREXAMPLE

Let M be a type II_∞ factor represented as $M_0 \otimes B(l^2(\mathbb{N}))$ where M_0 is a type II_1 factor with faithful normal trace τ . Fix $\varepsilon > 0$ and consider the operators

$$x_n = q \oplus \underbrace{\varepsilon p_1 \oplus \varepsilon p_1 \oplus \cdots \oplus \varepsilon p_1}_{n \text{ times}} \oplus p_2 \oplus p_3 \oplus \cdots,$$

$x_n \in M$, p_i projections in M_0 ; $p_1 \geq p_2 \geq \cdots$, $\tau(p_n) = 2^{-n}$, $\tau(q) > 0$.

Define S_ε to be the ultraweakly closed linear span of the operators x_n . S_ε is local since $x_n \in J(M)$ for every $n \geq 1$.

Lemma 2.1. *For every $x \in S_\varepsilon \cap J(M)$ there exist:*

- (i) *an increasing sequence of integers $1 \leq n_1 < n_2 < \cdots < n_k < \cdots$ and*
- (ii) *a convergence series $\sum_{k \geq 1} \lambda_k$ of complex numbers such that for every $k \geq 1$ there exists a net $(y_\alpha^k)_{\alpha \in A_k}$ of finite linear combinations of x_n 's, $n > n_k$, such that the net*

$$x_\alpha^k = \lambda_1 x_{n_1} + \cdots + \lambda_k x_{n_k} + y_\alpha^k$$

converges to x and A_{k+1} is a cofinal subset of A_k .

Moreover, for every $\delta > 0$ there is $k_0 = k_0(\delta)$ such that, if s_α^k denotes the sum of the coefficients in y_α^k , we have

$$|s_\alpha^k| < \delta \quad \text{for every } k \geq k_0, \\ \alpha \geq \alpha_k \in A_k \quad \text{for some } \alpha_k \in A_k.$$

Proof. Let $(\xi_n)_{n \geq 0}$ be an orthonormal basis in $l^2(\mathbb{N})$, and let E_n denote the projection onto the one-dimensional space generated by ξ_n . For any diagonal operator $x \in M \otimes B(l^2(\mathbb{N}))$, $(I \otimes E_n)x$ will be referred to as the n th component of x .

If $x = 0$, simply take $n_k = k$, $\lambda_k = 0$; hence, let us assume that x is different from 0.

Since $x \in S_\varepsilon$, there exists a net of finite linear combinations of x_n 's which converges ultraweakly to x . This net has the form

$$(\mu_1^\alpha x_1 + \mu_2^\alpha x_2 + \cdots + \mu_n^\alpha x_n + \cdots)_{\alpha \in A}$$

where all but finitely many μ_n^α 's are equal to 0. If $s_\alpha = \mu_1^\alpha + \mu_2^\alpha + \cdots + \mu_n^\alpha + \cdots$, the zeroth component of this net shows that $\lim_\alpha s_\alpha = s$ for some $s \in \mathbb{C}$.

If $m_\alpha = \min\{i; \mu_i^\alpha \neq 0\}$, we claim that $(m_\alpha)_{\alpha \in A}$ has a convergent subnet. Indeed, if some subnet would tend to $+\infty$, then for every $c \in \mathbb{N}$ choose α such that $m_\alpha > c$; hence,

$$(\mu_{m_\alpha}^\alpha x_{m_\alpha} + \mu_{m_\alpha+1}^\alpha x_{m_\alpha+1} + \cdots)_{\alpha \in B}$$

converges to x , where B is cofinal in A .

Considering the first c components of this net, notice that they are

$$s_\alpha q \oplus s_\alpha \varepsilon p_1 \oplus \cdots \oplus s_\alpha \varepsilon p_1.$$

Since $c \in \mathbb{N}$ was arbitrarily large, this implies that

$$x = sq \oplus s\varepsilon p_1 \oplus s\varepsilon p_1 \oplus \cdots.$$

Our assumption that $x \neq 0$ ensures $s \neq 0$, but this contradicts $x \in J(M)$.

The net $(m_\alpha)_{\alpha \in A}$ has a cluster point in $\overline{\mathbb{N}}$, the one-point compactification of $\mathbb{N}$. By the above remarks, this cluster point is different from $+\infty$.

Consequently, there exists an integer $n_1 \geq 1$ and a subnet $(\lambda_1^\alpha)_{\alpha \in A_1}$ such that

$$(\lambda_1^\alpha x_{n_1} + \mu_{n_1+1}^\alpha x_{n_1+1} + \cdots)_{\alpha \in A_1}$$

converges to x . If we look at the $(n_1 + 1)$ st component of this net and multiply it by any nonzero projection $e \leq p_1 - p_2$ we get $e(\mu_{n_1+1}^\alpha + \mu_{n_1+2}^\alpha + \cdots)$. Ultraweak convergence implies that

$$s_\alpha - \lambda_1^\alpha = \mu_{n_1+1}^\alpha + \mu_{n_1+2}^\alpha + \cdots$$

is convergent; hence, λ_1^α is a convergent net in $\mathbb{C}$. Suppose that $\lim_\alpha \lambda_1^\alpha = \lambda_1 \in \mathbb{C}$. It follows that

$$\lim_\alpha (\lambda_1 x_{n_1} + \mu_{n_1+1}^\alpha x_{n_1+1} + \cdots)_{\alpha \in A_1} = x,$$

that is, $\lim_\alpha (\lambda_1 x_{n_1} + y_\alpha^1)_{\alpha \in A_1} = x$, where A_1 is cofinal in A .

The construction continues by induction on k , after replacing x by $x - \lambda_1 x_{n_1} - \cdots - \lambda_{k-1} x_{n_{k-1}}$.

Let us prove the last assertion. Suppose that for some $\delta > 0$ there is a sequence $(k_i)_{i \geq 1}$ such that $|s_\alpha^{k_i}| \geq \delta$, $\alpha \in A'_{k_i}$ cofinal in A_{k_i} . For $0 \neq e \leq p_1 - p_2$ define

$$\begin{array}{ccccccc} \tilde{e} = 0 \oplus \cdots \oplus 0 & \oplus e \oplus 0 \oplus \cdots \oplus 0 & \oplus e \oplus 0 \oplus \cdots & & & & \\ & \downarrow & & \downarrow & & & \\ & n_1 + 1 & & n_2 + 1 & & & \end{array}$$

Let us now remark that for every $\alpha \in A_{k_j}$ and for every $j \geq 1$ the first $n_j + 1$ components of $\tilde{e}x_\alpha^j$ are

$$\begin{array}{ccccccc} 0 \oplus \cdots \oplus 0 \oplus es_\alpha^1 \oplus 0 \oplus \cdots \oplus 0 & \oplus es_\alpha^2 \oplus 0 \oplus \cdots \oplus 0 & \oplus es_\alpha^j & & & & \\ & \downarrow & & \downarrow & & \downarrow & \\ & n_1 + 1 & & n_2 + 1 & & n_j + 1 & \end{array}$$

By taking into account our assumption that infinitely many $s_\alpha^{k_i}$ satisfy $|s_\alpha^{k_i}| \geq \delta$, it follows that the operator $\tilde{e}x \in J(M)$ has infinitely many components of absolute value greater or equal to δe . This contradiction establishes our assertion and the fact that the series $\sum \lambda_k$ is convergent.

Notice that the operator $y_e = q \oplus \varepsilon p_1 \oplus \varepsilon p_1 \oplus \varepsilon p_1 \oplus \cdots$ belongs to S_ε .

Lemma 2.2. *If $\varepsilon = m^{-2}$ ($m \geq 6$) and $y = y_{m^{-2}}$ then $\text{dist}(y, J(M)) = m^{-2}$ and $\text{dist}(y, S_{m^{-2}} \cap J(M)) \geq m^{-1}$.*

Proof. By removing the zeroth component, namely, the one corresponding to $q \in J(M)$, one can easily see that

$$\text{dist}(y, J(M)) = \varepsilon = m^{-2}.$$

Suppose that $\text{dist}(y, S_{m^{-2}} \cap J(M)) < m^{-1}$, and choose $x \in S_{m^{-2}} \cap J(M)$ such that $\|y - x\| < m^{-1}$. With notation in Lemma 2.1 and its proof, it follows that $|1 - s| < m^{-1}$. Choose α_0 such that $\alpha \geq \alpha_0$ implies $|1 - s_\alpha| < m^{-1}$. By Lemma 2.1, choose $k \geq 1$ such that $|s_\alpha^k| < m^{-1}$, $\alpha \in A'_k$ cofinal in A_k , which entails $|1 - (\lambda_1 + \cdots + \lambda_k)| < 2m^{-1}$; hence, $|\lambda_1 + \cdots + \lambda_k| > 1 - 2m^{-1}$. The $(n_k + 1)$ st component multiplied by p_{n_k+1} shows that

$$\|y - x\| \geq |\varepsilon - (\lambda_1 + \cdots + \lambda_k) - \varepsilon s_\alpha^k| - m^{-1} \geq 1 - 3m^{-1} - m^{-2} - m^{-3} \geq 1 - 5m^{-1}$$

for α large enough. Since the inequality $1 - 5m^{-1} < m^{-1}$ fails to be true for $m \geq 6$, our assertion is proved.

Proposition 2.3. *Every type II_∞ factor contains a local subspace whose compact ideal perturbation is not norm-closed.*

Proof. If we define $S = \bigoplus_{m \geq 6} S_{m^{-2}}$, then S is local in $M \otimes B(l^2(\mathbb{N}))$, but the canonical algebraic isomorphism between $(S + J(M))/J(M)$ and $S/(S \cap J(M))$ fails to be a topological isomorphism, by Lemma 2.2. This proves that $S + J(M)$ is not norm-closed.

The general case is now immediate. Since $M = M_0 \otimes B(l^2(\mathbb{N}))$, note that $M_0 \otimes B(P_m l^2(\mathbb{N}))$ will contain a copy of $S_{m^{-2}}$, where P_m is the projection of $l^2(\mathbb{N})$ onto the subspace generated by $\{\xi_{p_m}, \xi_{p_m^2}, \xi_{p_m^3}, \dots\}$, p_m being the m th prime number. The projections P_m are mutually orthogonal, which concludes the proof.

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