UNIVALENT LOGHARMONIC RING MAPPINGS

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(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. Univalent logharmonic ring mappings are characterized in terms of univalent starlike mappings. An existence and uniqueness theorem is also given.

1. INTRODUCTION

Let D be a domain of $\mathbb{C}$ and denote by $H(D)$ the set of all analytic functions on D . A logharmonic mapping is a solution of the nonlinear elliptic partial differential equation

$$(1.1) \quad \overline{f_z} = (a\overline{f}/f)f_z,$$

where the second dilatation function a is in $H(D)$ and $|a(z)| < 1$ for all $z \in D$. If f does not vanish in D , then f is of the form

$$f = H\overline{G},$$

where H and G are locally analytic (possibly multivalued) functions on D . On the other hand, if f vanishes at z_0 but is not identically zero, then f admits the local representation

$$f(z) = (z - z_0)^m |z - z_0|^{2\beta} h(z) \overline{g(z)}$$

where $m \in \mathbb{N}$, $\operatorname{Re} \beta > -m/2$, and h and g are analytic in a neighbourhood of z_0 (see, e.g., [1]).

Univalent logharmonic mappings defined on the unit disk U have been studied in [1, 2].

In this paper we investigate the family $\mathcal{A}_r$ of all univalent logharmonic mappings which map an annulus $A(r, 1) = \{z : r < |z| < 1\}$, $0 < r < 1$, onto an annulus $A(R, 1)$ for some $R \in [0, 1)$ and which satisfy the condition $\frac{1}{2\pi} \int_{|z|=\rho} d \arg f(\rho e^{it}) = 1$ for all $\rho \in (r, 1)$. The last condition says that the outer boundary corresponds to the outer boundary. We call an element $f \in \mathcal{A}_r$ a univalent logharmonic ring mapping.

If $a \equiv 0$, then $R = r$ and $f(z) = e^{i\alpha} z$, $\alpha \in \mathbb{R}$, are the only mappings in $\mathcal{A}_r$. In the case of univalent harmonic mappings from $A(r, 1)$ onto $A(R, 1)$ it is possible that $R = 0$; for example, $[1/(1-r^2)][z - r^2/\overline{z}]$ has this property. However, Nitsche [11] has shown that there is an $R_0(r) < 1$ such that there is

Received by the editors February 27, 1991.

1991 *Mathematics Subject Classification*. Primary 30C55.

no univalent harmonic mapping from $A(r, 1)$ onto $A(R, 1)$ whenever $R_0 < R < 1$.

There is no univalent logharmonic mappings from $A(r, 1)$, $0 < r < 1$, onto $A(0, 1)$. This is a direct consequence of (2.1) in our Theorem 2.1. But, on the other hand, we have for R neither a positive lower bound nor a uniform upper bound strictly less than one. Indeed, $f(z) = z|z|^{2\beta}$, $\operatorname{Re} \beta > -\frac{1}{2}$, is univalent on $A(r, 1)$ and its image is $A(r^{1+2\operatorname{Re} \beta}, 1)$.

Unlike the case of univalent harmonic mappings, univalent logharmonic ring mappings need not have a continuous extension onto the closure of $A(r, 1)$. Indeed,

$$f(z) = z(\overline{z-1})/(1-z)$$

is a univalent logharmonic ring mapping from $A(\frac{1}{2}, 1)$ onto itself whose cluster sets on the outer boundary are $C(f, e^{it}) = \{-1\}$, if $z = e^{it}$, $0 < t < 2\pi$, and $C(f, 1) = \{w; |w| = 1\}$.

Theorems 2.1 and 2.2 give a complete characterization of univalent logharmonic mappings in $\mathcal{A}_r$. It turns out that these are exactly the mappings of the form

$$f(z) = \left(\frac{\varphi(z)}{|\varphi(z)|} |z|^{2\gamma} \right),$$

where $\operatorname{Re} \gamma > 0$, $\varphi \in H(A(r, 1))$, and where $p(z) = z\varphi'(z)/\varphi(z) = 1 + \sum_{k \in \mathbb{Z} \setminus \{0\}} p_k z^k$ satisfies $\operatorname{Re} p(z) > 0$ on $A(r, 1)$. It follows that functions in $\mathcal{A}_r$ map concentric circles onto concentric circles. Such a question was raised by Nitsche [12] for univalent harmonic ring mappings and was negatively answered in [6].

Theorem 2.3 is an existence and uniqueness result for a fixed given dilatation function a . This is possible, since the cluster set of a boundary point of $A(r, 1)$ is either a singleton or an interval on the circle $|w| = R$ (resp. $|w| = 1$).

We shall use the following notation. Denote by $\mathcal{P}_r$ the class

$$(1.2) \quad \mathcal{P}_r = \left\{ p \in H(A(r, 1)); \operatorname{Re} p > 0 \text{ on } A(r, 1) \text{ and } \frac{1}{2\pi i} \oint_{|z|=\rho} p(z) \frac{dz}{z} = 1 \text{ and for some } \rho \in (r, 1) \right\}.$$

This class has been studied by several authors, for example, in [3, 7–10, 13]. In particular, mappings in $\mathcal{P}_r$ can be represented by the generalized Herglotz formula

$$(1.3) \quad p(z) = \int_{-\pi}^{\pi} p^*(ze^{-it}) d\mu(t) + \int_{-\pi}^{\pi} p^*\left(\frac{r}{ze^{-it}}\right) d\nu(t) - 1$$

where

$$(1.4) \quad p^*(z) = \frac{1+z}{1-z} + 2 \sum_{k=1}^{\infty} \frac{r^{2k}}{1-r^{2k}} (z^k - z^{-k})$$

and where μ and ν are probability measures on the Borel σ -algebra over $[-\pi, \pi)$. Note that (1.3) reduces to the classical Herglotz formula if $r = 0$. The mapping p^* is univalent on $A(r, 1)$ and maps $A(r, 1)$ onto the right half-plane cut along a vertical slit, which lies on $\{w; \operatorname{Re} w \equiv 1\}$ and is bisected by the real axis.

Let $S^*(r, 1)$ be the set of all univalent analytic functions φ on $A(r, 1)$ having the properties:

- (i) $p(z) = z\varphi'(z)/\varphi(z) \in H(A(r, 1))$,
- (ii) $\operatorname{Re} p(z) > 0$ on $A(r, 1)$.

The condition of φ to be univalent can be replaced by the property $p(z) = 1 + \sum_{k \in \mathbb{Z} \setminus \{0\}} p_k z^k$. For completeness we give a short proof.

Lemma 1.1. $\varphi \in S^*(r, 1)$ if and only if $p(z) = z\varphi'(z)/\varphi(z) \in \mathcal{P}_r$.

Proof. If $\varphi \in S^*(r, 1)$, it is sufficient to show that $p_0 = 1$. Indeed, we have for $\rho \in (r, 1)$

$$p_0 = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{z\varphi'(z)}{\varphi(z)} \frac{dz}{z} = \frac{1}{2\pi} \oint_{|z|=\rho} d \arg \varphi(\rho e^{it}) = 1.$$

Next, suppose that $p(z) = z\varphi'(z)/\varphi(z) = 1 + \sum_{k \in \mathbb{Z} \setminus \{0\}} p_k z^k \in H(A(r, 1))$ and that $\operatorname{Re} p(z) > 0$ on $A(r, 1)$. Then $\varphi \in H(A(r, 1))$. We show that φ is univalent on $A(r, 1)$. Fix $\rho \in (r, 1)$. Since $\operatorname{Re} p(z) > 0$ on $A(r, 1)$ and $p_0 = 1$, we have

$$\frac{1}{2\pi} \oint_{|z|=\rho} d \arg z\varphi'(z) = \frac{1}{2\pi} \oint_{|z|=\rho} d \arg \varphi(z) = p_0 = 1.$$

Moreover, we conclude from the relation

$$\frac{\partial \arg \varphi(\rho e^{it})}{\partial t} = \operatorname{Re} \left\{ \frac{\rho e^{it} \varphi'(\rho e^{it})}{\varphi(\rho e^{it})} \right\} > 0$$

that φ is univalent on $\{z : |z| = \rho\}$ and that $\Gamma_\rho = \{w : w = \varphi(\rho e^{it}); 0 \leq t \leq 2\pi\}$ is a simple closed strictly starlike curve winding once around the origin. Finally, for $r < \rho_1 < \rho_2 < 1$, we have $\Gamma_{\rho_1} \cap \Gamma_{\rho_2} = \emptyset$. Indeed, $\varphi(\rho_1 < |z| < \rho_2)$ has to be a bounded domain. Therefore, φ is univalent on $\{z : \rho_1 < |z| < \rho_2\}$ for all $r < \rho_1 < \rho_2 < 1$ and, hence, also on $A(r, 1)$.

Let f be in $\mathcal{A}_r$. Then $F(\zeta) = \log f(e^\zeta)$ is a univalent harmonic mapping from the vertical strip $\mathcal{S}_c = \{\zeta : -c < \operatorname{Re} \zeta < 0\}$, $c = -\ln r$, onto the vertical strip $S_{-\log R}$. Furthermore, F is a solution of the elliptic partial differential equation

$$(1.5) \quad \overline{F_\zeta} = a F_\zeta,$$

where the second dilatation function a satisfies

$$(1.6) \quad a \in H(\mathcal{S}_c), \quad |a| < 1, \quad \text{and} \quad a(\zeta + 2\pi i) = a(\zeta) \text{ on } \mathcal{S}_c.$$

Such mappings F appear in a natural way in studying the nonparametric minimal surfaces over a vertical strip whose normal directions are periodic. (For more details see [4, 12].)

Suppose now that a satisfies (1.6) and that $F = U + iV$ is a univalent solution of (1.5) which maps $\mathcal{S}_c$ onto $\mathcal{S}_d$ for some $d > 0$ such that $\lim_{\xi \uparrow 0} U(\xi + i\eta) \equiv 0$ and $\lim_{\xi \downarrow -c} U(\xi + i\eta) \equiv -d$. We show in Theorem 3.1 that there is a $\delta > 0$ such that the function $f(z) = \exp[\frac{2\pi}{\delta} F(\log z)]$ belongs to $\mathcal{A}_r$.

In what follows we shall frequently use the expression $|z|^{2\beta}$. We always take the branch corresponding to $1^{2\beta} = 1$.

2. LOGHARMONIC RING MAPPINGS

We start with a characterization of univalent logharmonic ring mappings.

Theorem 2.1. *A function f belongs to $\mathcal{A}_r$ if and only if*

$$(2.1) \quad f(z) = z|z|^{2\beta}h(z)/\overline{h(z)},$$

where

- (a) $h \in H(A(r, 1))$ and $0 \notin h(A(r, 1))$;
- (b) $\operatorname{Re}(zh'(z)/h(z)) > -1/2$ on $A(r, 1)$;
- (c) $\frac{1}{2\pi} \int_{|z|=\rho} d \arg h(z) = 0$, $r < \rho < 1$;
- (d) $\operatorname{Re} \beta > -1/2$.

In particular, functions belonging to $\mathcal{A}_r$ map concentric circles onto concentric circles.

Proof. Let $f \in \mathcal{A}_r$, and suppose that $f(A(r, 1)) = A(R, 1)$. Then we have $f(z) = H(z)\overline{G(z)}$ where H and G are locally analytic (possibly multivalued) functions on $A(r, 1)$. Furthermore, because $f(z) \neq 0$ on $A(r, 1)$, the functions

$$\frac{f_z}{f} = \frac{H'}{H} = \sum_{k \in \mathbb{Z}} a_k z^k, \quad \frac{\overline{f_z}}{f} = \frac{G'}{G} = \sum_{k \in \mathbb{Z}} b_k z^k$$

are analytic on $A(r, 1)$. By integration and exponentiation we get formally

$$f(z) = z^{a-1} \overline{z^{b-1}} h(z) \overline{g(z)}$$

where h and g belong to $H(A(r, 1))$ and satisfy

$$(2.2) \quad \frac{1}{2\pi} \oint_{|z|=\rho} d \arg h(z) = \frac{1}{2\pi} \oint_{|z|=\rho} d \arg g(z) = 0$$

for all $\rho \in (r, 1)$. Therefore, the conditions (a) and (c) hold.

The identity $f(ze^{2\pi i}) \equiv f(z)$ implies that $e^{2\pi i(a-1-\overline{b-1})} = 1$ and that $a-1-\overline{b-1} = n \in \mathbb{Z}$. So far, we have the representation

$$f(z) = z^n |z|^{2\beta} h(z) \overline{g(z)}, \quad n \in \mathbb{Z}, \quad \beta = \overline{b-1}.$$

Since

$$(2.3) \quad |h\overline{g}|_{|z|=r} \equiv R/r^{n+2\operatorname{Re} \beta} \quad \text{and} \quad |h\overline{g}|_{|z|=1} \equiv 1,$$

we conclude that $hg = e^{i\alpha} z^m$, $\alpha \in \mathbb{R}$, $m \in \mathbb{Z}$. But (2.2) shows that $m = 0$, and we are lead to the representation

$$f(z) = e^{i\alpha} z^n |z|^{2\beta} \frac{h(z)}{\overline{h(z)}}, \quad h \in H(A(r, 1)).$$

Finally, we have

$$(2.4) \quad 1 = \oint_{|z|=\rho} d \arg f(z) = n + 2 \oint_{|z|=\rho} d \arg h(z) = n$$

for all $\rho \in (r, 1)$, from which it follows that $n = 1$. Replacing $h(z)$ by $e^{-i\alpha/2}h(z)$, we get the representation formula

$$f(z) = z|z|^{2\beta}h(z)/\overline{h(z)}.$$

The relation (2.1) implies that $|f(re^{i\theta})| = R = r^{1+2\operatorname{Re}\beta}$, and the relation $\operatorname{Re}\beta = (1/2)[(\log R)/\log r - 1] > -1/2$ shows that (d) holds. In order to prove (b), first observe that $|f(z)|_{|z|=\rho} = \rho^{1+2\operatorname{Re}\beta}$, $\rho \in (r, 1)$. Since f is univalent on $A(r, 1)$ and $\oint_{|z|=\rho} d\arg f(z) = 1$, we get for $z = \rho e^{it} \in A(r, 1)$

$$\begin{aligned} (2.5) \quad \frac{\partial \arg f(\rho e^{it})}{\partial t} &= \operatorname{Im} \left(i \frac{zf_z}{f} - i \frac{\bar{z}\bar{f}_{\bar{z}}}{\bar{f}} \right) = \operatorname{Re} \left(\frac{zf_z}{f} - \frac{\bar{z}\bar{f}_{\bar{z}}}{\bar{f}} \right) \\ &= \operatorname{Re} \left(z \left(\frac{1+\beta}{z} + \frac{h'}{h} - \frac{\beta}{z} + \frac{h'}{h} \right) \right) = \operatorname{Re} \left(1 + 2 \frac{zh'(z)}{h(z)} \right) > 0, \end{aligned}$$

and (b) is proved. The necessity part is established.

Conversely, assume that $f(z) = z|z|^{2\beta}h(z)/\overline{h(z)}$ and that the conditions (a)–(d) are satisfied. We have to prove that f is a univalent logharmonic mapping from $A(r, 1)$ onto $A(R, 1)$ for some $R \in (0, 1)$ and that $\oint_{|z|=\rho} d\arg f(z) = 1$ for all $\rho \in (r, 1)$. First of all, we see that f maps any circle $\{|z| = \rho\}$, $\rho \in (r, 1)$, into the circle $\{|w| = \rho^{1+2\operatorname{Re}\beta}\}$. Let us show that f is a solution of (1.1). By (2.1), we have

$$a = \frac{\bar{f}_{\bar{z}}/\bar{f}}{f_z/f} = \frac{\bar{\beta} - zh'/h}{1 + \beta + zh'/h} \in H(A(r, 1)),$$

and, since $\operatorname{Re}\beta > -1/2$ and $\operatorname{Re}(zh'(z)/h(z)) > -1/2$ on $A(r, 1)$, elementary calculations show that $|a| < 1$ on $A(r, 1)$. Next, we show that f is univalent on $A(r, 1)$. Using the relation (2.5) and condition (b), we conclude that each circle $\{z : |z| = \rho\}$, $\rho \in (r, 1)$, is mapped monotonically onto the circle $\{w : |w| = \rho^{1+2\operatorname{Re}\beta}\}$, and, from condition (c), we deduce that f is univalent in $A(r, 1)$ and that $\oint_{|z|=\rho} d\arg f = 1$. Therefore, $f \in \mathcal{A}_r$. This completes the proof.

Another version of Theorem 2.1 is the following representation.

Theorem 2.2. *A function f is in $\mathcal{A}_r$ if and only if it is of the form*

$$(2.6) \quad f(z) = \left(\frac{\varphi(z)}{|\varphi(z)|} |z|^{2\gamma} \right),$$

where $\operatorname{Re}\gamma > 0$ and $\varphi \in S^*(r, 1)$.

Proof. Let f be in $\mathcal{A}_r$. Then, by Theorem 2.1, we have

$$f(z) = z|z|^{2\beta}h(z)/\overline{h(z)},$$

where $\operatorname{Re}\beta > -1/2$, $\operatorname{Re}(zh'(z)/h(z)) > -1/2$ on $A(r, 1)$, and $\int_{|z|=\rho} d\arg h(z) = 0$ for all $\rho \in (r, 1)$. Put $\varphi(z) = zh^2(z)$. It follows that

$$(2.7) \quad f(z) = z|z|^{2\beta} \frac{h^2(z)}{|h(z)|^2} = \left(\frac{\varphi(z)}{|\varphi(z)|} |z|^{2\beta+1} \right) = \left(\frac{\varphi(z)}{|\varphi(z)|} |z|^{2\gamma} \right),$$

where $\operatorname{Re} \gamma > 0$. On the other hand, we have

$$(2.8) \quad \operatorname{Re}(z\varphi'(z)/\varphi(z)) = 1 + 2\operatorname{Re}(zh'(z)/h(z)) > 0$$

for all $z \in A(r, 1)$. Moreover, putting $p(z) = z\varphi'(z)/\varphi(z)$ we get

$$(2.9) \quad \begin{aligned} p_0 &= \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{p(z)}{z} dz = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{\varphi'(z)}{\varphi(z)} dz = \frac{1}{2\pi} \oint_{|z|=\rho} d\arg \varphi(z) \\ &= 1 + \frac{1}{\pi} \oint_{|z|=\rho} d\arg h(z) = 1, \end{aligned}$$

which implies (Lemma 1.1) that $\varphi \in S^*(r, 1)$.

Conversely, let $f = |z|^{2\gamma}\varphi/|\varphi|$, $\operatorname{Re} \gamma > 0$, and $\varphi \in S^*(r, 1)$. We have to show that $h(z) = \sqrt{\varphi(z)/z}$ can be defined as a single-valued function on $A(r, 1)$. Observe first that

$$\frac{zh'}{h} = \frac{[z\varphi'/\varphi - 1]}{2} \in H(A(r, 1)).$$

By (2.9), we have

$$\frac{1}{2\pi} \oint_{|z|=\rho} d\arg h = \frac{1}{2} \left\{ \left[\frac{1}{2\pi} \oint_{|z|=\rho} d\arg \varphi \right] - 1 \right\} = \frac{p_0 - 1}{2} = 0.$$

Therefore, we can define $\log h$ and, hence, also h as an analytic function on $A(r, 1)$. Evidently, h does not vanish on $A(r, 1)$; so we have shown properties (a) and (c) of h in Theorem 2.1. The properties (b) and (d) of h and the form (2.1) of f follow from the relations (2.7) and (2.8). Finally, we apply Theorem 2.1 and the proof is complete.

For the next result, we fix the second dilatation function $a \in H(A(r, 1))$, $|a(z)| < 1$ for all $z \in A(r, 1)$.

Theorem 2.3. *For a given $a \in H(A(r, 1))$, $|a(z)| < 1$ for all $z \in A(r, 1)$ and for a given $z_0 \in A(r, 1)$, there exists one and only one univalent solution f of (1.1) in $\mathcal{A}_r$ such that $f(z_0) > 0$.*

Remark. Theorem 2.3 is not true for univalent harmonic ring mappings (see [6, Theorem 7.3]).

In the proof of Theorem 2.3 we shall need the following lemma.

Lemma 2.4. *There exists a β depending on a such that $\operatorname{Re} \beta > -1/2$ and*

$$I = \int_0^{2\pi} \frac{\bar{\beta} - (1 + \beta)a(\rho e^{it})}{1 + a(\rho e^{it})} dt = 0$$

for all $\rho \in (r, 1)$.

Proof. Observe that I does not depend on ρ since the integrand is analytic on $A(r, 1)$. Furthermore (writing $a(\rho e^{it}) = a$), the condition

$$(2.10) \quad I = \int_0^{2\pi} \frac{\bar{\beta} - |a|^2 - \beta a - a + \bar{a}\bar{\beta} - \beta|a|^2}{|1 + a|^2} dt = 0$$

is satisfied for

$$\operatorname{Re} \beta = \left(\int_0^{2\pi} \frac{\operatorname{Re} a + |a|^2}{|1 + a|^2} dt \right) \left(\int_0^{2\pi} \frac{1 - |a|^2}{|1 + a|^2} dt \right)^{-1}$$

and

$$\operatorname{Im} \beta = -\frac{(1 + 2 \operatorname{Re} \beta)}{2\pi} \int_0^{2\pi} \frac{\operatorname{Im} a}{|1 + a|^2} dt.$$

Furthermore, we have

$$1 + 2 \operatorname{Re} \beta = 2\pi \left(\int_0^{2\pi} \frac{1 - |a|^2}{|1 + a|^2} dt \right)^{-1} > 0,$$

and the lemma is shown.

Proof of Theorem 2.3. Consider the function

$$h(z) = \exp \left(\int \frac{\bar{\beta} - (1 + \beta)a(z)}{z(1 + a(z))} dz \right),$$

where β is defined in Lemma 2.4. Since

$$\frac{1}{2\pi} \oint_{|z|=\rho} d \arg h(z) = \frac{1}{2\pi i} \oint_{|z|=\rho} \frac{\bar{\beta} - (1 + \beta)a(z)}{z(1 + a(z))} dz = \frac{I}{2\pi} = 0,$$

it follows that $h \in H(A(r, 1))$ and that

$$\operatorname{Re} \left(\frac{zh'(z)}{h(z)} + \frac{1}{2} \right) = \frac{1 - |a|^2}{|1 + a|^2} \left(\frac{1}{2} + \operatorname{Re} \beta \right) > 0 \quad \text{on } A(r, 1).$$

This implies that $\operatorname{Re}(zh'(z)/h(z)) > -1/2$ on $A(r, 1)$. Applying Theorem 2.1, we conclude that the function $\hat{f}$ defined by

$$\hat{f}(z) = z|z|^{2\beta} h(z)/\overline{h(z)}$$

belongs to $\mathcal{A}_r$ and is a solution of (1.1). Finally, take

$$f(z) = (\hat{f}(z_0)/|\hat{f}(z_0)|)\hat{f}(z),$$

and the proof of the existence is established.

We now show the uniqueness of the mapping. Suppose that f_1 and f_2 are two solutions of (1.1) satisfying the conditions of Theorem 2.3.

First, we show that $f_1(z_0) = f_2(z_0)$, which implies that $\operatorname{Re} \beta$ is unique, since $f_1(z_0) = f_2(z_0) = r^{1+2\operatorname{Re} \beta}$.

Suppose that the contrary holds. Put

$$\frac{f_1(z_0)}{f_2(z_0)} = |z_0|^{2\operatorname{Re}(\beta_1 - \beta_2)} = s > 0 \quad \text{and} \quad \Phi(z) = s \cdot \frac{f_2(z)}{f_1(z)}.$$

The function Φ is again a nonconstant solution of (1.1) with respect to the same dilatation function a and is hence an orientation-preserving open mapping. We have $|\Phi(\rho e^{it})| = (\rho/|z_0|)^{2\operatorname{Re}(\beta_2 - \beta_1)}$ for all $\rho \in (r, 1)$. With no loss of generality, we may assume that $\operatorname{Re}(\beta_2 - \beta_1) > 0$; if not, consider $1/\Phi$. Then we have $|\Phi(e^{it})| > 1$ and $|\Phi(re^{it})| < 1$. Choose $r < \rho_1 < \rho_2 < 1$ such that $|\Phi(\rho_1 e^{it})| < 1$ and $|\Phi(\rho_2 e^{it})| > 1$. Then, by the argument principle, we have

$$J = \frac{1}{2\pi} \oint_{|z|=\rho_2} d \arg(\Phi(z) - 1) - \frac{1}{2\pi} \oint_{|z|=\rho_1} d \arg(\Phi(z) - 1) \geq 1,$$

since $\rho_1 < |z_0| < \rho_2$ and $\Phi(z_0) = 1$. On the other hand,

$$\frac{1}{2\pi} \oint_{|z|=\rho_2} d \arg(f_1) - \frac{1}{2\pi} \oint_{|z|=\rho_1} d \arg(f_1) = 0$$

and, therefore,

$$J = \frac{1}{2\pi} \oint_{|z|=\rho_2} d \arg(sf_2 - f_1) - \frac{1}{2\pi} \oint_{|z|=\rho_1} d \arg(sf_2 - f_1).$$

Since, for $|z_0| \neq \rho$,

$$s|f_2(\rho e^{it})| - |f_1(\rho e^{it})| = \rho^{1+2\operatorname{Re}(\beta_1)} \left(\left(\frac{|z_0|}{\rho} \right)^{2\operatorname{Re}(\beta_1 - \beta_2)} - 1 \right) \neq 0,$$

we conclude that $J = 0$, which leads to a contradiction. Hence, $f_1(z_0) = f_2(z_0)$, which implies that $|\phi| \equiv 1$ on $A(r, 1)$ and $\Phi(z_0) = 1$. Since Φ is a solution of (1.1) but is not an open mapping, it follows that $\Phi \equiv 1$, and, therefore, $f_1 \equiv f_2$.

The next result gives an algebraic structure of the class $\mathcal{A}_r$.

Theorem 2.5. *The family $\mathcal{A}_r$ is “logarithmically convex”, i.e., if $f_1, f_2 \in \mathcal{A}_r$, then $F = f_1^\lambda f_2^{1-\lambda} \in \mathcal{A}_r$ for all $\lambda \in (0, 1)$.*

Proof. Let φ_j , $j = 1, 2$, be the univalent starlike mappings defined in Theorem 2.2 corresponding to f_j , $j = 1, 2$. Then the functions $p_j(z) = z\varphi'_j(z)/\varphi_j(z)$ belong to $\mathcal{P}_r$. Since $p = \lambda p_1 + (1 - \lambda)p_2$ has the same property, we conclude from Lemma 1.1 that $\varphi = \varphi_1^\lambda \varphi_2^{1-\lambda} \in S^*(r, 1)$. Finally, by Theorem 2.2, we have

$$f(z) = f_1^\lambda(z) \cdot f_2^{1-\lambda}(z) = \left(\frac{\varphi(z)}{|\varphi(z)|} |z|^{2(\lambda\gamma_1 + (1-\lambda)\gamma_2)} \right) = \left(\frac{\varphi(z)}{|\varphi(z)|} |z|^{2\gamma} \right)$$

for all $z \in A(r, 1)$, and, therefore, f belongs to $\mathcal{A}_r$.

3. UNIVALENT HARMONIC STRIP MAPPINGS WITH PERIODIC DILATATIONS

Denote by $\mathcal{S}_c$ the vertical strip $\{\zeta; -c < \operatorname{Re} \zeta < 0\}$, $\zeta = \xi + i\eta$, and let Σ_c be the set of all univalent harmonic and orientation-preserving mappings $F = U + iV$ from $\mathcal{S}_c$ onto $\mathcal{S}_d$ for some $d > 0$ such that $\lim_{\xi \uparrow 0} U(\xi + i\eta) \equiv 0$ and $\lim_{\xi \downarrow -c} U(\xi + i\eta) \equiv -d$. It follows that $U(\zeta) = d\xi/c$ and that there is an $a \in H(\mathcal{S}_c)$, $|a| < 1$ on $\mathcal{S}_c$, such that F is a solution of the elliptic partial differential equation

$$(3.1) \quad \overline{F_\zeta} = aF_\zeta.$$

Furthermore, F admits the representation

$$(3.2) \quad F(\zeta) = \frac{d}{c} \left[\xi + i \operatorname{Im} \int^\zeta \frac{1 - a(s)}{1 + a(s)} ds \right].$$

Indeed, (3.1) is equivalent to the equation

$$iV_\zeta = \frac{1 - a}{1 + a} U_\zeta.$$

Let $\mathcal{F}_c$ be the set of all mappings F which are of the form (3.2). First, we observe that each $F \in \mathcal{F}_c$ is univalent harmonic and orientation-preserving on $\mathcal{S}_c$. Evidently, F is harmonic, and we conclude from

$$\overline{F_\zeta} = \frac{d}{c} \cdot \frac{a}{1+a} \quad \text{and} \quad F_\zeta = \frac{d}{c} \cdot \frac{1}{1+a}$$

that F is a solution of (3.1) and, therefore, is orientation-preserving. The univalence of F follows from the fact that the vertical lines $\{\zeta : \operatorname{Re} \zeta = \xi_0\}$, $\xi_0 \in (-c, 0)$, are mapped into the vertical lines $\{w : \operatorname{Re} w = d\xi_0/c\}$ and that $\frac{\partial V}{\partial \eta} = \operatorname{Re} \frac{(1-a)}{1+a} > 0$ on $\mathcal{S}_c$.

There are mappings in $\mathcal{F}_c$ which do not belong to Σ_c . For instance,

$$F(\zeta) = \zeta - \frac{i}{2} e^{2\eta} \sin(2\xi)$$

belongs to $\mathcal{F}_{\pi/2}$ but not to $\Sigma_{\pi/2}$. Indeed, F is of the form (3.2) with

$$a(s) = \frac{1 + ie^{-2is}}{1 - ie^{-2is}}, \quad s \in \mathcal{S}_{\pi/2},$$

and

$$F(\mathcal{S}_{\pi/2}) = \{w; -\pi/2 < \operatorname{Re} w < 0 \text{ and } \operatorname{Im} w > 0\}$$

is not a vertical strip. On the other hand, by similar arguments as in [5, Theorem 2.7], Σ_c is dense in $\mathcal{F}_c$ with respect to the topology of the locally uniform convergence.

Suppose now that a satisfies the periodicity condition

$$a(\zeta + 2\pi i) = a(\zeta), \quad \zeta \text{ in } \mathcal{S}_c.$$

Then (3.2) defines a univalent solution of (3.1), which maps $\mathcal{S}_c$ into $\mathcal{S}_d$. Let us show that $F = U + iV \in \Sigma_c$, i.e., that F maps $\mathcal{S}_c$ onto $\mathcal{S}_d$. Indeed, $F(\zeta + 2\pi i)$ is also a univalent solution of (3.1), and, since

$$F(\zeta + 2\pi i) - F(\zeta) = \frac{d}{c} i \operatorname{Im} \int_{\zeta}^{\zeta+2\pi i} \frac{1 - a(s)}{1 + a(s)} ds$$

is not an open mapping, it follows that

$$(3.3) \quad F(\zeta + 2\pi i) \equiv F(\zeta) + i\delta \quad \text{on } \mathcal{S}_c$$

for some $\delta \in \mathcal{R}$. The univalence of F and the property $\partial V / \partial \eta > 0$ imply that $\delta > 0$, from which we conclude that the vertical lines $\{\zeta : \operatorname{Re} \zeta = \xi_0\}$, $\xi_0 \in (-c, 0)$, are mapped onto the vertical lines $\{w : \operatorname{Re} w = d\xi_0/c\}$.

Summarizing we have shown:

Theorem 3.1. *Let $a \in H(\mathcal{S}_c)$, $|a| < 1$, and $a(\zeta + 2\pi i) = a(\zeta)$ on $\mathcal{S}_c$. Then (3.2) defines a univalent solution F of (3.1) which belongs to Σ_c . Furthermore, any solution of (3.1) which belongs to Σ_c is of the form (3.2), and there is a $\delta > 0$ such that (3.3) holds.*

Define

$$\Sigma_{c,\mathcal{P}} = \{F \in \Sigma_c; a(\zeta + 2\pi i) = a(\zeta) \text{ on } \mathcal{S}_c\}.$$

For $F \in \Sigma_{c, \mathcal{P}}$, consider the mapping

$$(3.4) \quad f(z) = \exp \left[\frac{2\pi}{\delta} F(\log z) \right],$$

where δ is defined in (3.3). Then f is a univalent and logharmonic mapping on $A(e^{-c}, 1)$ and belongs to the class $\mathcal{A}_r$, $r = e^{-c}$. Conversely, if $f \in \mathcal{A}_r$ then $F(\zeta) = \log f(e^\zeta) \in \Sigma_{c, \mathcal{P}}$, $c = -\log r$. It is easy to transfer results from one class to the other. For instance, we may get a second proof of the existence in Theorem 2.3 as follows:

Let $a_1 \in H(A(r, 1))$, $|a_1| < 1$ on $A(r, 1)$, and let $z_0 \in A(r, 1)$ be given. Put $a(s) = a_1(e^s)$, and let F be defined by (3.2) and f by (3.4). Then $(\overline{f(z_0)}/|f(z_0)|) \cdot f$ is the desired solution.

Observe that $\Sigma_{c, \mathcal{P}}$ and $\mathcal{A}_r$ are not compact families with respect to the topology of locally uniform convergence. However, we may still estimate some functionals over the classes $\Sigma_{c, \mathcal{P}}$ and $\mathcal{A}_r$, since $\mathcal{P}_r$ is compact. For instance, the problem

$$M(\rho, r) = \max_{f \in \mathcal{P}^*(r, 1)} \left\{ \max_{0 \leq t \leq 2\pi} \left\{ |\arg e^{-it} f(\rho e^{it})| \bmod 2\pi \right\} \right\}, \quad r < \rho < 1,$$

has been solved in [3]. Using Theorem 2.2, we get immediately that

$$|\arg e^{-it} f(\rho e^{it})|_{\bmod 2\pi} \leq M(r, \rho) + 2 |\operatorname{Im} \gamma| \ln \rho$$

for all $f \in \mathcal{A}_r$.

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