BEST APPROXIMATIONS IN L^1 ARE NEAR BEST IN L^p , $p < 1$

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ABSTRACT. We show that any best L^1 polynomial approximation to a function f in L^p , $0 < p < 1$, is near best in L^p .

Let $I = [0, 1]^d$ and let $\mathcal{P}_r$ be the set of all polynomials in d variables of total degree less than r . It is known that for each f in $L^1(I)$ a (not necessarily unique) best approximation $E_1 f$ to f exists in $\mathcal{P}_r$ and that $E_1 f = q$ if and only if

$$(1) \quad \left| \int_{E_+} s(x) dx - \int_{E_-} s(x) dx \right| \leq \int_{E_0} |s(x)| dx \quad \text{for all } s \text{ in } \mathcal{P}_r,$$

where $E_+ = \{x \in I \mid q(x) > f(x)\}$, $E_- = \{x \in I \mid q(x) < f(x)\}$, and $E_0 = \{x \in I \mid q(x) = f(x)\}$. Condition (1) makes sense even if f is not in $L^1(I)$, and when (1) is satisfied, we call q a best $L^1(I)$ approximation to f and denote q by $E_1 f$. It is easy to show that for constant approximations ($r = 1$) the extended E_1 , which is the median operator, is defined for all measurable f and bounded on $L^p(I)$ for any $p > 0$ (see [2] for a discussion of medians). It is also easy to show that the similarly extended L^2 best projection operator onto polynomials is bounded on L^p for $p \geq 1$ and any $r > 0$. These facts motivate us to prove the following theorem.

Theorem. For each f in $L^p(I)$, $0 < p < 1$, and for all $r > 0$, a best $L^1(I)$ approximation $E_1 f$ exists in $\mathcal{P}_r$. Moreover, for all choices of $E_1 f$,

$$\|f - E_1 f\|_p \leq (1 + 2K)^{1/p} \inf_{q \in \mathcal{P}_r} \|f - q\|_p,$$

where

$$K = \sup_{q \in \mathcal{P}_r} \frac{\|q\|_\infty}{\|q\|_1}.$$

This theorem provides a method to find near-best polynomial approximations in $L^p(I)$ for $0 < p < 1$. Such approximations are useful in atomic decompositions of the Besov spaces $B_p^\alpha(L^p(I))$, $p < 1$, which are the regularity spaces

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for nonlinear approximation in $L^q(I)$, $q^{-1} = p^{-1} - \alpha/d$, by wavelets and free-knot splines. Theory and applications to image and surface compression can be found in [2–6]. Of course, such near-best approximations are known to exist; the new thing here is that $L^1(I)$ projections provide them.

We make several remarks about the theorem. First, a more careful argument shows that

$$\|f - E_1 f\|_p \leq (2K)^{1/p-1} \inf_{q \in \mathcal{P}_r} \|f - q\|_p$$

for $0 < p < 1$; see [1]. Second, our proof extends to approximation by any finite-dimensional subspace of $L^1(\Omega) \cap L^\infty(\Omega)$ for any finite measure space $(\Omega, d\mu)$. (A measure space with atoms must first be embedded into a continuous measure space.) Generalizations to the behavior of the best $L^p(I)$ approximation in $L^{p-1}(I)$ for $p > 1$ and the fact that the best $L^1(I)$ approximation is defined for any measurable f are contained in [1].

We first prove the following lemma, which contains the main argument.

Lemma. *If there exists a best $L^1(I)$ approximation $E_1 f \in \mathcal{P}_r$ to $f \in L^p(I)$, then*

$$\|E_1 f\|_p \leq \left(2 \sup_{q \in \mathcal{P}_r} \frac{\|q\|_\infty}{\|q\|_1} \right)^{1/p} \|f\|_p.$$

Proof. Condition (1) implies, in particular, that for $q = E_1 f$,

$$(2) \quad \left| \int_{E_+} q(x) dx - \int_{E_-} q(x) dx \right| \leq \int_{E_0} |q(x)| dx.$$

Our approach is to bound $\|f\|_p$ from below among all f satisfying (2) for a particular q .

We introduce the function

$$g(x) = \begin{cases} 0, & x \in E_+, \quad q(x) > 0, \\ q(x), & x \in E_+, \quad q(x) \leq 0, \\ 0, & x \in E_-, \quad q(x) < 0, \\ q(x), & x \in E_-, \quad q(x) \geq 0, \\ q(x), & x \in E_0. \end{cases}$$

Clearly, $\|g\|_p \leq \|f\|_p$ and the sets $\tilde{E}_+ \subset E_+$, $\tilde{E}_- \subset E_-$, and $\tilde{E}_0 \supset E_0$ for g and q satisfy

$$(3) \quad \left| \int_{\tilde{E}_+} q(x) dx - \int_{\tilde{E}_-} q(x) dx \right| \leq \int_{\tilde{E}_0} |q(x)| dx.$$

Now $\int_I |g|^p = \int_{\tilde{E}_0} |q|^p$, and to further bound $\|f\|_p$ from below we minimize $\int_{\tilde{E}_0} |q(x)|^p dx$ among all partitions $(\tilde{E}_+, \tilde{E}_-, \tilde{E}_0)$ with $q > 0$ on $\tilde{E}_+$, $q < 0$ on $\tilde{E}_-$, and (3) holding. These conditions imply that

$$\int_{\tilde{E}_+ \cup \tilde{E}_-} |q(x)| dx \leq \int_{\tilde{E}_0} |q(x)| dx;$$

i.e.,

$$\int_{\tilde{E}_0} |q(x)| dx \geq \frac{1}{2} \|q\|_1.$$

We claim that the best choice of $\tilde{E}_0$ satisfies

$$\inf_{x \in \tilde{E}_0} |q(x)| \geq \sup_{x \notin \tilde{E}_0} |q(x)| \quad \text{and} \quad \int_{\tilde{E}_0} |q(x)| dx = \frac{1}{2} \|q\|_1.$$

Suppose $\tilde{E}'_0$ is any other choice. We can assume $\int_{\tilde{E}'_0} |q(x)| dx = \frac{1}{2} \|q\|_1$, because otherwise we could make $\tilde{E}'_0$, and, *a fortiori*, $\int_{\tilde{E}'_0} |q(x)|^p dx$, smaller.

Let a be any number between $\sup_{x \notin \tilde{E}_0} |q(x)|$ and $\inf_{x \in \tilde{E}_0} |q(x)|$, and let $A = \tilde{E}_0 \setminus \tilde{E}'_0$ and $B = \tilde{E}'_0 \setminus \tilde{E}_0$. Then $\int_A |q(x)| dx = \int_B |q(x)| dx$ and

$$\begin{aligned} \int_A |q(x)| dx &= \int_A |q(x)|^p |q(x)|^{1-p} dx \geq a^{1-p} \int_A |q(x)|^p dx, \\ \int_B |q(x)| dx &= \int_B |q(x)|^p |q(x)|^{1-p} dx \leq a^{1-p} \int_B |q(x)|^p dx. \end{aligned}$$

Therefore,

$$\int_A |q(x)|^p dx \leq \int_B |q(x)|^p dx$$

and

$$\int_{\tilde{E}_0} |q(x)|^p dx \leq \int_{\tilde{E}'_0} |q(x)|^p dx.$$

So

$$\frac{\int_I |q(x)|^p dx}{\int_I |f(x)|^p dx} \leq \frac{\int_I |q(x)|^p dx}{\int_{\tilde{E}_0} |q(x)|^p dx}.$$

Since $\int_{\tilde{E}_0} |q(x)| dx = \frac{1}{2} \|q\|_1$, we have

$$\|q\|_\infty |\tilde{E}_0| \geq \frac{1}{2} \|q\|_1 \quad \text{or} \quad |\tilde{E}_0| \geq \frac{1}{2} \frac{\|q\|_1}{\|q\|_\infty}.$$

Now

$$\int_{I \setminus \tilde{E}_0} |q(x)|^p dx \leq a^p (1 - |\tilde{E}_0|)$$

and

$$\int_{\tilde{E}_0} |q(x)|^p dx \geq a^p |\tilde{E}_0|.$$

So

$$\frac{\int_I |q(x)|^p dx}{\int_{\tilde{E}_0} |q(x)|^p dx} = 1 + \frac{\int_{I \setminus \tilde{E}_0} |q(x)|^p dx}{\int_{\tilde{E}_0} |q(x)|^p dx} \leq 1 + \frac{1 - |\tilde{E}_0|}{|\tilde{E}_0|} = \frac{1}{|\tilde{E}_0|} \leq 2 \frac{\|q\|_\infty}{\|q\|_1}.$$

Therefore, $\|q\|_p \leq (2K)^{1/p} \|f\|_p$, where $K = \sup_{q \in \mathcal{P}_r} (\|q\|_\infty / \|q\|_1)$. $\square$

Corollary. For each f in $L^p(I)$, $0 < p < 1$, a best $L^1(I)$ polynomial approximation $E_1 f$ exists.

Proof. For each positive integer n , define f_n by

$$f_n(x) = \begin{cases} n, & f(x) > n, \\ f(x), & |f(x)| \leq n, \\ -n, & f(x) < -n. \end{cases}$$

Then $f_n \in L^\infty(I)$ and $\|f_n\|_p \leq \|f\|_p$. Best $L^1(I)$ approximations q_n to f_n exist for all n and

$$\|q_n\|_p \leq C\|f_n\|_p \leq C\|f\|_p.$$

However, for each p and r there is a constant C_1 such that for all $q \in \mathcal{P}_r$, $\|q\|_\infty \leq C_1\|q\|_p$. Therefore for all n we have $\|q_n\|_\infty \leq CC_1\|f\|_p$, so that for some n we have $\|q_n\|_\infty < n$. This implies that we can choose $E_1 f = q_n$ because the sets E_+ , E_- , and E_0 are the same for f as for f_n . $\square$

Proof of the main theorem. The corollary shows that $E_1 f$ exists. Now E_1 is linear with respect to addition of polynomials in $\mathcal{P}_r$: If we let $g = f + q$, $q \in \mathcal{P}_r$, and $E_1 g = E_1 f + q$, then clearly condition (1) is satisfied because E_+ , E_- , and E_0 are the same for g and $E_1 g$ as for f and $E_1 f$. So $E_1 f + q$ is a best $L^1(I)$ approximation to $f + q$ according to our definition.

Finally, because $\|f + g\|_p^p \leq \|f\|_p^p + \|g\|_p^p$, we have for all $q \in \mathcal{P}_r$,

$$\begin{aligned} \|f - E_1 f\|_p^p &\leq \|f - q\|_p^p + \|q - E_1 f\|_p^p \\ &= \|f - q\|_p^p + \|E_1(q - f)\|_p^p \\ &\leq (1 + 2K)\|f - q\|_p^p \end{aligned}$$

by the lemma. $\square$

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