CLASSIFYING PL 5-MANIFOLDS UP TO REGULAR GENUS SEVEN

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ABSTRACT. In the present paper, we show that the only closed orientable 5-manifolds of regular genus less or equal than seven are the 5-sphere S^5 and the connected sums of m copies of $S^1 \times S^4$, with $m \leq 7$. As a consequence, the genus of $S^3 \times S^2$ is proved to be eight. This suggests a possible approach to the (3-dimensional) Poincaré Conjecture, via the well-known classification of simply connected 5-manifolds, obtained by Smale and Barden.

1. STATEMENTS

The possibility of representing a large class of polyhedra, including all PL-manifolds of dimension n , by means of $(n+1)$ -coloured graphs is well known (see, as general references, [FGG, BM, V]).

Within this representation theory, a nonnegative combinatorial invariant $\mathcal{G}(M^n)$, called the regular genus of M^n , has been introduced for each n -manifold M^n . It extends to dimension n the classical notions of genus of a surface and of Heegaard genus of a 3-manifold [G2]; it further characterizes the n -sphere S^n among all closed n -manifolds, as follows [FG]:

$$\mathcal{G}(M^n) = 0 \quad \text{iff} \quad M^n \cong S^n$$

(here “ $\cong$ ” means PL-homeomorphism).

In the present paper, we obtain the classification of all closed connected orientable 5-dimensional PL-manifolds with regular genus less than or equal to 7:

Main Theorem. *Let M^5 be a closed connected orientable PL 5-manifold. Then*

$$1 \leq \mathcal{G}(M^5) = m \leq 7 \quad \text{iff} \quad M^5 \cong \#_m(S^1 \times S^4)$$

(where $\#_m(S^1 \times S^4)$ denotes the connected sum of m copies of $S^1 \times S^4$).

As a direct consequence, we have

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Corollary I. (a) $\mathcal{G}(\mathbb{S}^3 \times \mathbb{S}^2) = 8$.

(b) For every closed orientable PL 3-manifold M^3 , $\mathcal{G}(M^3 \times \mathbb{S}^2) \geq 8$.

The proof of the above results will be presented in §3.

Remark A. By [B], if Σ^3 is a homotopy 3-sphere, then $\Sigma^3 \times \mathbb{S}^2 \cong \mathbb{S}^3 \times \mathbb{S}^2$; therefore, by Corollary I(a), $\mathcal{G}(\Sigma^3 \times \mathbb{S}^2) = 8$. We believe that the above relation characterizes homotopy 3-spheres among orientable 3-manifolds (i.e., that $\mathcal{G}(M^3 \times \mathbb{S}^2) = 8$ iff M^3 is a homotopy 3-sphere). On the other hand, since Corollary I gives a positive answer to parts (a) and (b') of Conjecture I of [GG] (see also [G3]), we can state the following stronger

Conjecture I'. Let M^3 be any closed orientable 3-manifold. Then

$$\mathcal{G}(M^3 \times \mathbb{S}^2) = 8 \quad \text{iff} \quad M^3 \cong \mathbb{S}^3.$$

Conjecture I' seems, of course, of hard solution: in fact, we recall that it actually implies the (3-dimensional) Poincaré Conjecture, by the classification of simply connected 5-manifolds, obtained in [S, B]. Let us propose the following general problem about the genus of product manifolds, presented as a sequence of conjectures, for the sake of conciseness.

Conjecture II_n. For every closed connected orientable 3-manifold M^3 , $\mathcal{G}(M^3 \times \mathbb{S}^n) \geq \mathcal{G}(M^3 \times \mathbb{D}^n)$ (where $\mathbb{D}^n$ denotes the n -disk).

Remark B. Conjectures II₁ and II₂ together imply the Poincaré Conjecture for all closed 3-manifolds M^3 of Heegaard genus $\mathcal{H}(M^3) \geq 5$.

In fact, by [CP], $\mathcal{G}(M^h) \geq \mathcal{G}(\partial M^h)$, where M^h is any h -manifold with boundary ∂M^h and $\mathcal{G}(\partial M^h)$ denotes the sum of the genera of its components. Hence, assuming Conjecture II₂, $\mathcal{G}(M^3 \times \mathbb{S}^2) \geq \mathcal{G}(M^3 \times \mathbb{D}^2) \geq \mathcal{G}(M^3 \times \mathbb{S}^1)$; furthermore, assuming Conjecture II₁, $\mathcal{G}(M^3 \times \mathbb{S}^1) \geq \mathcal{G}(M^3 \times \mathbb{D}^1) \geq 2\mathcal{G}(M^3) = 2\mathcal{H}(M^3)$.

This proves that, if $\mathcal{H}(M^3) \geq 5$, then $\mathcal{G}(M^3 \times \mathbb{S}^2) \geq 2\mathcal{H}(M^3) \geq 10$, and therefore M^3 is not a homotopy sphere, since for each homotopy 3-sphere Σ^3 we have $\mathcal{G}(\Sigma^3 \times \mathbb{S}^2) = \mathcal{G}(\mathbb{S}^3 \times \mathbb{S}^2) = 8$, by Corollary I.

2. PRELIMINARIES AND NOTATIONS

In this paper, we shall work with manifolds (closed, connected, and orientable, unless otherwise stated) and maps in the PL-category, for which we refer to [Gl] or [RS]; we will often identify a ball-complex K with its associated polyhedron and every homeomorphic space. For graph theory, see [W].

An $(n+1)$ -coloured graph is a pair (Γ, γ) , where $\Gamma = (V(\Gamma), E(\Gamma))$ is a multigraph (i.e., multiple edges are allowed, but loops are forbidden), regular of degree $n+1$, and $\gamma: E(\Gamma) \rightarrow \Delta_n = \{i \in \mathbb{Z} | 0 \leq i \leq n\}$ is a proper edge-colouring on Γ (i.e., $\gamma(e) \neq \gamma(f)$ for any two adjacent edges $e, f \in E(\Gamma)$). For the sake of conciseness, we shall often denote the $(n+1)$ -coloured graph (Γ, γ) simply by the symbol Γ of its underlying multigraph.

For each $\mathcal{B} \subseteq \Delta_n$, set $\Gamma_{\mathcal{B}} = (V(\Gamma), \gamma^{-1}(\mathcal{B}))$; each connected component of $\Gamma_{\mathcal{B}}$ is called a $\mathcal{B}$ -residue, or an m -residue if the cardinality $\#\mathcal{B}$ of $\mathcal{B}$ is m . For each $i \in \Delta_n$, set $\hat{i} = \Delta_n - \{i\}$. The symbol $\mathbf{g}(\Gamma)$ (resp. $\mathbf{g}_{\mathcal{B}} = \mathbf{g}_{\mathcal{B}}(\Gamma)$) will denote the number of components of Γ (resp. of $\mathcal{B}$ -residues of Γ); if $\mathcal{B} = \{i, j\}$ (resp. $\mathcal{B} = \{i, j, k\}$), we shall write $\mathbf{g}_{ij}$ (resp. $\mathbf{g}_{ijk}$) instead of $\mathbf{g}_{\mathcal{B}}$ and $\mathbf{g}_{ij}$ (resp. $\mathbf{g}_{ijk}$) instead of $\mathbf{g}_{\Delta_n - \mathcal{B}}$.

An n -dimensional labelled pseudocomplex $K(\Gamma)$ (see [HW, BM]) can be associated to every $(n+1)$ -coloured graph (Γ, γ) ; the construction, described in [FGG], works as follows:

- (i) for each $\mathbf{v} \in V(\Gamma)$, take an n -simplex $\sigma(\mathbf{v})$, with its vertices labelled $0, 1, \dots, n$;
- (ii) for each $\mathbf{e} \in E(\Gamma)$ with end points $\mathbf{v}, \mathbf{w} \in V(\Gamma)$ and colour $\gamma(\mathbf{e}) = i$, identify the $(n-1)$ -faces of $\sigma(\mathbf{v})$ and $\sigma(\mathbf{w})$ opposite of the vertex labelled i .

We say that (Γ, γ) represents $K(\Gamma)$ and every homeomorphic polyhedron. It is easy to check that, for each $\mathcal{B} \in \Delta_n$, with $\#\mathcal{B} = h \leq n$, there is a bijection between the set of $\mathcal{B}$ -residues of Γ and the set of $(n-h)$ -simplices of $K(\Gamma)$, whose vertices are labelled $\Delta_n - \mathcal{B}$. Moreover, it is known that:

- $K(\Gamma)$ is orientable iff Γ is a bipartite graph;
- $K(\Gamma)$ is a manifold iff, for every n -residue Θ of Γ , $K(\Theta)$ is homeomorphic to the $(n-1)$ -sphere $\mathbb{S}^{n-1}$.

A *crystallization* of a closed connected n -manifold M^n is any $(n+1)$ -coloured graph (Γ, γ) representing M^n , such that Γ_{ε} is connected for each colour $c \in \Delta_n$. The basic results of [P1, P2] ensure the existence of crystallizations for every closed n -manifold. An alternative proof is contained in [LM].

In [G2] the existence is proved, for each connected bipartite $(n+1)$ -coloured graph (Γ, γ) of order $\#V(\Gamma) = \mathbf{p}$ and for each cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ of Δ_n , of a particular 2-cell embedding (see [W]) $\iota: \Gamma \rightarrow F_{\varepsilon}$, where F_{ε} denotes the orientable closed surface of genus $\rho(F_{\varepsilon}) = \rho_{\varepsilon}(\Gamma) = 1 - \chi_{\varepsilon}(\Gamma)/2$, where

$$\chi_{\varepsilon}(\Gamma) = \sum_{c \in \mathbb{Z}_{n+1}} \mathbf{g}_{\varepsilon_c \varepsilon_{c+1}} + (1-n)\mathbf{p}/2.$$

The nonnegative integers $\rho(\Gamma) = \min_{\varepsilon} \{\rho_{\varepsilon}(\Gamma)\}$ and $\mathcal{G}(M^n) = \min \{\rho(\Gamma) | \Gamma \text{ is a crystallization of } M^n\}$ are respectively called the *regular genus* of the graph Γ and of the manifold M^n .

From now on, (Γ, γ) is assumed to be a bipartite $(n+1)$ -coloured graph representing a (possibly disconnected) closed orientable n -manifold. If Γ (resp. $\Gamma_{\mathcal{B}}$, with $\mathcal{B} \subseteq \Delta_n$) has $\mathbf{g} = \mathbf{g}(\Gamma) \geq 1$ (resp. $\mathbf{g}_{\mathcal{B}} = \mathbf{g}_{\mathcal{B}}(\Gamma) \geq 1$) connected components $\Gamma^{(1)}, \Gamma^{(2)}, \dots, \Gamma^{(\mathbf{g})}$ (resp. $\Xi^{(1)}, \Xi^{(2)}, \dots, \Xi^{(\mathbf{g}_{\mathcal{B}})}$), we define $\rho = \rho_{\varepsilon}(\Gamma) = \sum_r \rho_{\varepsilon}(\Gamma^{(r)})$ (resp. $\rho_{\mathcal{B}} = \rho_{\varepsilon'}(\Gamma_{\mathcal{B}}) = \sum_r \rho_{\varepsilon'}(\Xi^{(r)})$), where $\rho_{\varepsilon}(\Gamma^{(r)})$ (resp. $\rho_{\varepsilon'}(\Xi^{(r)})$) denotes the genus of $\Gamma^{(r)}$ (resp. of $\Xi^{(r)}$), with respect to the cyclic permutation $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ of Δ_n (resp. to the cyclic permutation ε' induced by ε on the subset $\mathcal{B}$ of Δ_n). For the sake of notational simplicity, we shall always write “ i ” instead of “ ε_i ”, for all $i \in \Delta_n$. Hence, the following formulae, written for the fundamental cyclic permutation $(0, 1, \dots, n)$, actually hold for every cyclic permutation $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n)$ of Δ_n .

Now, the relation $\chi_{\varepsilon}(\Gamma) = 2\mathbf{g}(\Gamma) - 2\rho_{\varepsilon}(\Gamma) = 2\mathbf{g} - 2\rho$ gives

$$(1) \quad \sum_{c \in \mathbb{Z}_{n+1}} \mathbf{g}_{c, c+1} + (1-n)\mathbf{p}/2 = 2\mathbf{g} - 2\rho.$$

For each subgraph Γ_i , with $i \in \Delta_n$, the following analogous formula holds:

$$(1_i) \quad \mathbf{g}_{i-1, i+1} + \sum_{c \in \mathbb{Z}_{n+1}^{(i)}} \mathbf{g}_{c, c+1} + (2-n)\mathbf{p}/2 = 2\mathbf{g}_i - 2\rho_i,$$

where $\mathbb{Z}_{n+1}^{(i)}$ denotes the set $\Delta_n - \{i-1, i\}$ and the sum is mod $n+1$.

By substituting (1_i) into (1) and by making use of the relation

$$(2) \quad 2g_{ijk} = g_{ij} + g_{jk} + g_{ki} - \mathbf{p}/2$$

(which is a direct consequence of $\chi(K(\Theta)) = \chi(\mathbb{S}^2) = 2$, for each $\{i, j, k\}$ -residue Θ), we obtain (for details, see [GG, Lemmas 6.1 and 6.2])

$$(3_i) \quad g_{i-1, i+1} = g_{i-1, i} + \rho - \rho_i + g_i - g.$$

The following formula is simply obtained by taking the difference between relation (3_i), applied to the graph (Γ, γ) , and the same relation, applied to the partial graph Γ_j , with $j \neq i \pm 1$:

$$(4) \quad g_{ij} = g_i + g_j - g + \rho - \rho_i - \rho_j + \rho_{ij} \quad (j \neq i \pm 1)$$

where we have set $\rho_{ij} = \rho_{\Delta_n - \{i, j\}}$.

Remark C. Since $g - g_i \geq g_j - g_{ij}$ (as it is easy to check), formula (4) implies that, for every i, j nonconsecutive in ε , $\rho - \rho_i \geq \rho_j - \rho_{ij}$.

Let us now restrict our attention to the case of (Γ, γ) being a crystallization of a closed connected orientable 5-manifold M^5 . So, $g = g_c = 1$ for every $c \in \Delta_5$; moreover, if we set $\rho_{ijk} = \rho_{\Delta_n - \{i, j, k\}}$, then $\rho_{ijk} = 0$ for every $i, j, k \in \Delta_5$, since each 3-residue of Γ represents a 2-sphere. Hence, relation (4), applied to the graph Γ and to the subgroup Γ_k respectively, gives

$$(5) \quad g_{ij} = 1 + \rho - \rho_i - \rho_j + \rho_{ij}, \quad \text{for every } i, j \text{ nonconsecutive in } \varepsilon$$

and

$$(6) \quad g_{ijk} = g_{ik} + g_{jk} - 1 + \rho_k - \rho_{ik} - \rho_{jk}, \quad \text{for every } i, j \text{ nonconsecutive} \\ \text{in the cyclic permutation } \varepsilon_k \\ \text{induced by } \varepsilon \text{ on } \Delta_5 - \{k\}.$$

Remark D. Since, as in Remark C, $g_k - g_{ik} = 1 - g_{ik} \geq g_{jk} - g_{ijk}$, formula (6) actually implies that, for every i, j nonconsecutive in ε_k , $\rho_k - \rho_{ik} - \rho_{jk} \geq 0$.

By adding formulae (3_i) for $i \in \Delta_5$, i even (resp. i odd), and by making use of relation (2), the following formula (7) (resp. (8)) is obtained:

$$(7) \quad g_{024} = 1 + 2\rho - (\rho_0 + \rho_2 + \rho_4),$$

$$(8) \quad g_{135} = 1 + 2\rho - (\rho_1 + \rho_3 + \rho_5).$$

Moreover, since the subgraph Γ_i represents a four-dimensional sphere, the direct computation of $\chi(K(\Gamma_i))$ by means of formulae (1), (1_i), and (2) gives

$$(9) \quad \rho_i = \left(\sum_{j \neq i} \rho_{ij} \right) / 2$$

(see [GG, Proposition 6.3] for a detailed proof, in a more general setting).

3. CLASSIFYING 5-MANIFOLDS OF GENUS ≤ 7

In the present section, (Γ, γ) is assumed to be a crystallization of a closed connected orientable 5-manifold M^5 . If $K = K(\Gamma)$ denotes the triangulation of

M^5 associated to Γ and $\{v_c | c \in \Delta_5\}$ is the vertex-set of K , then we may suppose that v_c corresponds to Γ_c , for each colour $c \in \Delta_5$. For each subset $\{i, j\}$ (resp. $\{i, j, k\}$) of distinct elements of Δ_5 , let $K(i, j)$ (resp. $K(i, j, k)$) be the subcomplex of K generated by the vertices v_i and v_j (resp. v_i, v_j , and v_k), and let $K(\hat{i}, \hat{j})$ (resp. $K(\hat{i}, \hat{j}, \hat{k})$) be the subcomplex of K generated by the vertex-set $\{v_c | c \in \Delta_5 - \{i, j\}\}$ (resp. $\{v_d | d \in \Delta_5 - \{i, j, k\}\}$). Let further $N(i, j)$, $N(i, j, k)$, $N(\hat{i}, \hat{j})$, and $N(\hat{i}, \hat{j}, \hat{k})$ be the regular neighbourhood in K of $K(i, j)$, $K(i, j, k)$, $K(\hat{i}, \hat{j})$, and $K(\hat{i}, \hat{j}, \hat{k})$ respectively.

The following general results are needed in the proofs of the Main Theorem and of Corollary 1.

Lemma 1a [C]. *Let H be a two-dimensional pseudocomplex which contains exactly three vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$. For each pair $\mathbf{v}, \mathbf{w}$ of distinct vertices of H , let $H(\mathbf{v}, \mathbf{w})$ be the subcomplex of H , generated by the vertex-set $\{\mathbf{v}, \mathbf{w}\}$. Let further $[\mathbf{v}, \mathbf{w}]$ (resp. $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$) denote the number of edges of $H(\mathbf{v}, \mathbf{w})$ (resp. the number of triangles of H). If H satisfies the following properties:*

- (i) $1 \leq [\mathbf{a}, \mathbf{c}] \leq [\mathbf{a}, \mathbf{b}] - 1$;
- (ii) $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{a}, \mathbf{b}] + [\mathbf{a}, \mathbf{c}] - 1$;
- (iii) *the inclusion $j: H(\mathbf{b}, \mathbf{c}) \rightarrow H$ induces an epimorphism $j_*: \pi_1(H(\mathbf{b}, \mathbf{c})) \rightarrow \pi_1(H)$,*

then H collapses to $H(\mathbf{b}, \mathbf{c})$.

Proof. First of all, we note that property (ii) implies the following property:

- (iv) all edges of $H(\mathbf{a}, \mathbf{b})$ are faces of at least one triangle of H .

In fact, if an edge e of $H(\mathbf{a}, \mathbf{b})$ is not a face of any triangle of H , then any edge-loop in H containing e could not be homotopic to an edge-loop of $H(\mathbf{b}, \mathbf{c})$; this obviously contradicts (iii), and hence (iv) holds, too. As a consequence of properties (i), (ii), (iv), only $r \leq [\mathbf{a}, \mathbf{c}] - 1 \leq [\mathbf{a}, \mathbf{b}] - 2$ edges $(\mathbf{a}, \mathbf{b})$ are faces of more than one triangle of H ; then, some triangles of H collapse from their edges “ $(\mathbf{a}, \mathbf{b})$ ” on their edges “ $(\mathbf{a}, \mathbf{c})$ ” and “ $(\mathbf{b}, \mathbf{c})$ ”. Hence, H collapses to a subcomplex H' such that $H'(\mathbf{b}, \mathbf{c}) = H(\mathbf{b}, \mathbf{c})$. If $r = 0$, it is easy to see that $[\mathbf{a}, \mathbf{c}]$ must be equal to 1; that is, H' is a graph collapsing to $H'(\mathbf{b}, \mathbf{c})$. On the other hand, if $r \geq 1$, then H' satisfies again the properties (i), (ii), and (iii), with the vertices $\mathbf{b}$ and $\mathbf{c}$ exchanged of meaning. Thus, the process may be iterated, until $r = 0$, leaving $H(\mathbf{b}, \mathbf{c})$ fixed. $\square$

Lemma 1b. *Let H be a two-dimensional pseudocomplex which contains exactly three vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$. If H satisfies the following properties:*

- (i) $[\mathbf{a}, \mathbf{b}] = [\mathbf{a}, \mathbf{c}] = 3$;
- (ii) $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}] + 3$;
- (iii) *for $\mathbf{z} \in \{\mathbf{b}, \mathbf{c}\}$, the inclusion $j_z: H(\mathbf{a}, \mathbf{z}) \rightarrow H$ induces an epimorphism $j_{z*}: \pi_1(H(\mathbf{a}, \mathbf{z})) \rightarrow \pi_1(H)$;*
- (iv) $H_1(H; \mathbb{Z})$ is free and $H_2(H; \mathbb{Z}) = 0$,

then H collapses to a graph.

Proof. First of all we note, as in the proof of Lemma 1a, that property (iii) implies the following property, for every pair α, β of vertices of H :

- (v $_{\alpha\beta}$) all edges of $H(\alpha, \beta)$ are faces of at least one triangle of H .

Let now $\mathbf{e}_1, \dots, \mathbf{e}_h$ ($1 \leq h \leq 3$) be the h edges of $H(\mathbf{b}, \mathbf{c})$, which are faces of at least two triangles of H , and let $T_1, \dots, T_{3+h}$ be the $3+h$ triangles which

have an edge e_r as face, for $r = 1, \dots, h$. By property $(v_{b,c})$, H collapses to the subcomplex $H' = H(a, b) \cup H(a, c) \cup \{e_1, \dots, e_h\} \cup \{T_1, \dots, T_{3+h}\}$, with $1 \leq h \leq 3$, which of course satisfies all conditions (i), (ii), (iii), (iv) and, therefore, also $(v_{\alpha\beta})$, for every $\alpha, \beta \in \{a, b, c\}$. Moreover, property (iv) avoids the existence of any closed surface as a subcomplex of H' ; hence, the edges of $H'(a, b)$ and $H'(a, c)$ cannot be faces of exactly two triangles each. Since $[a, b] = [a, c] = 3 \geq (3+h)/2$, some triangles of H' collapse from their edges $“(a, x)”$, where either $x = b$ or $x = c$; w.l.o.g., $x = b$ may be assumed.

Note that every sequence of elementary collapses of triangles from their edges $“(a, b)”$ or $“(b, c)”$ produces a new complex H'' , satisfying property (iv) (since H' collapses to H'') and property (iii), only for $z = c$ (since $H''(a, c) = H'(a, c)$); hence H'' also satisfies properties (v_{ab}) and (v_{bc}) .

By looking in details to the three possible cases for the number h of edges of $H'(b, c)$, we can easily obtain such a complex H'' , with s edges $“(a, b)”$, s edges $“(b, c)”$, and $2s$ triangles, for $s = 1, 2$. By property (iv), H'' actually collapses to a graph. $\square$

By 5-dimensional handlebody of genus $m \geq 0$ we mean the orientable 5-manifold Y_m , which admits a decomposition with exactly the 0-handle and m 1-handles. Note that $Y_0 \cong \mathbb{D}^5$ (the 5-disk) and that, for $m \geq 1$, $\partial Y_m \cong \#_m(S^1 \times S^3)$.

Moreover, a 5-manifold with boundary M^5 is a handlebody (of genus m) iff it collapses to a graph Ω (with $\pi_1(\Omega) \cong {}_m\mathbb{Z}$).

Lemma 2. *Let (Γ, γ) be a crystallization of a closed connected orientable 5-manifold M^5 , and let $\varepsilon = (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_5)$ be a fixed permutation of Δ_5 . If there exists $\{r, s, t\} \subset \Delta_5$, with s, t nonconsecutive in ε_r , such that*

$$(*) \quad \rho_r = \rho_{rs} + \rho_{rt},$$

then $N(r, s, t)$ is a 5-dimensional handlebody Y_m of genus m , with $\text{rk}(\pi_1(M^5)) \leq m \leq \rho$.

Proof. Set $\alpha = g_{rsi} - g_{rs} \geq 0$; this means that there are α triangles in $K(r, s, t)$ more than edges in $K(r, s)$, i.e., $h(1 \leq h \leq \alpha)$ edges $e_1, \dots, e_h$ of $K(r, s)$ are faces of at least two triangles of $K(r, s, t)$. Let $T_1, \dots, T_{\alpha+h}$ be the $\alpha + h$ triangles of $K(r, s, t)$, which have an edge $e_i \in K(r, s)$ as face; it is easy to check that $K(r, s, t)$ collapses to the subcomplex $\bar{K} = K(r, t) \cup K(s, t) \cup \{e_1, \dots, e_h\} \cup \{T_1, \dots, T_{\alpha+h}\}$. By using $(*)$ into formula (6), $g_{rst} = g_{rs} + g_{rt} - 1$ is obtained.

Thus, the two-dimensional pseudocomplex $\bar{K}$ has: $[v_r, v_t] = 1 + \alpha$, $[v_r, v_s] = h \leq \alpha$, $[v_r, v_s, v_t] = \alpha + h$ (for the notation, see Lemma 1a). Moreover, since $K(\Gamma)$ is a pseudocomplex triangulating a 5-manifold and having exactly six vertices, the inclusion $j: K(s, t) \rightarrow K(r, s, t)$ induces an epimorphism $j_*: \pi_1(K(r, s)) \rightarrow \pi_1(K(r, s, t))$ [G1, Proposition 3]. The same property holds for $\bar{K}$ and $\bar{K}(r, s)$, too, since $\bar{K}$ is obtained from $K(r, s, t)$ by elementary collapses fixing $K(r, s)$. Hence, $\bar{K}$ satisfies all the hypothesis of Lemma 1a and collapses to $K(s, t)$. This obviously ensures that $N(r, s, t)$ is a 5-dimensional handlebody, of genus $m = g_{sti} - 1 \geq 0$. Moreover, since s and t are surely nonconsecutive in ε , [CP, Corollary 4.4] implies that $m = g_{sti} - 1 \leq \rho - \max\{\rho_s, \rho_t\} \leq \rho$.

The proof is now completed by recalling that the $(\Delta_5 - \{s, t\})$ -residues of Γ , but one arbitrarily chosen, are in bijection with the generators of a suitable presentation of $\pi_1(M^5)$ [G1]. $\square$

Proposition 3. *Let M^5 , (Γ, γ) and ε be as in Lemma 2. If there exists $\{i, j\} \subset \Delta_5$, such that $\rho_{ij} = 0$, then either $M^5 \cong \mathbb{S}^5$ or $M^5 \cong \#_m(\mathbb{S}^1 \times \mathbb{S}^4)$, $1 \leq m \leq \rho$.*

Proof. Let $\sigma = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ be the cyclic permutation induced by ε on $\Delta_5 - \{i, j\}$. Formula (9) directly implies $\rho_i = \rho_{i\hat{\sigma}_0} + \rho_{i\hat{\sigma}_2} (= \rho_{i\hat{\sigma}_1} + \rho_{i\hat{\sigma}_3})$ and $\rho_j = \rho_{j\hat{\sigma}_1} + \rho_{j\hat{\sigma}_3} (= \rho_{j\hat{\sigma}_0} + \rho_{j\hat{\sigma}_2})$. Since σ_0, σ_2 (resp. σ_1, σ_3) are nonconsecutive in ε_i (resp. ε_j), then Lemma 2 ensures that both $N(i, \sigma_0, \sigma_2)$ and $N(j, \sigma_1, \sigma_3)$ are 5-dimensional handlebodies, obviously of the same genus m , with $0 \leq m \leq \rho$. If $m = 0$, then $M^5 \cong \mathbb{S}^5$ directly follows. If $1 \leq m \leq \rho$, then $\partial N(i, \sigma_0, \sigma_2) \cong \partial N(j, \sigma_1, \sigma_3) \cong \#_m(\mathbb{S}^1 \times \mathbb{S}^3)$; so, the result is a consequence of Corollary 4.4 and Theorem 2.1 of [CH]. $\square$

Corollary 4. *Let M^5 , (Γ, γ) and ε be as in Lemma 2. If there exists $i \in \Delta_5$, such that $\rho_i \leq 2$, then either $M^5 \cong \mathbb{S}^5$ or $M^5 \cong \#_m(\mathbb{S}^1 \times \mathbb{S}^4)$, $1 \leq m \leq \rho$.*

Proof. As a consequence of formula (9) and Remark D, $\rho_i \leq 2$ implies the existence of $j \in \Delta_5 - \{i\}$, such that $\rho_{ij} = 0$. Thus, the thesis follows from Proposition 3. $\square$

Lemma 5. *Let M^5 , (Γ, γ) and ε be as in Lemma 2. If M^5 is simply connected and there exists $\{r, s, t\} \subset \Delta_5$, such that $N(r, s, t) \cong \mathbb{D}^5$, then $M^5 \cong \mathbb{S}^5$.*

Proof. The Mayer-Vietoris sequence of the pair $(N(r, s, t), N(\hat{r}, \hat{s}, \hat{t}))$ (with coefficients in $\mathbb{Z}$), together with Poincaré duality, easily gives $H_4(M^5) = H_3(M^5) = H_1(M^5) = 0$ and $\text{Tor } H_2(M^5) \cong H_2(M^5) \cong H_2(N(\hat{r}, \hat{s}, \hat{t}))$. On the other hand, $H_2(N(\hat{r}, \hat{s}, \hat{t}))$ is torsion-free ($K(\hat{r}, \hat{s}, \hat{t})$ being a two-dimensional pseudocomplex); hence, $H_2(M^5) = H_2(N(\hat{r}, \hat{s}, \hat{t})) = 0$. Now, since $H_*(M^5) = H_*(\mathbb{S}^5)$, the well-known Whitehead Theorem ensures that M^5 has the same homotopy type as $\mathbb{S}^5$; thus, the statement follows by the Generalized Poincaré Theorem. $\square$

Lemma 6. *Let M^5 , (Γ, γ) , and ε be as in Lemma 2. If there exists $\{r, s, t\} \subset \Delta_5$, such that $N(r, s, t) \cong Y_m$ where Y_m denotes a 5-dimensional handlebody of genus $m \geq 1$, then*

$$H_*(N(\hat{r}, \hat{s}, \hat{t}); \mathbb{Z}) \cong H_*(Y_m; \mathbb{Z}) \quad \text{and} \quad H_*(M^5; \mathbb{Z}) \cong H_*\left(\#_m(\mathbb{S}^1 \times \mathbb{S}^4); \mathbb{Z}\right).$$

Proof. Again, the Mayer-Vietoris sequence of the pair $(N(r, s, t), N(\hat{r}, \hat{s}, \hat{t}))$ (with coefficients in $\mathbb{Z}$), together with Poincaré duality easily gives: $H_5(M^5) \cong \mathbb{Z}$, $H_4(M^5) \cong H_1(M^5) \cong \bigoplus_m \mathbb{Z}$, $H_3(M^5) = 0$, and $\text{Tor } H_2(M^5) \cong H_2(M^5) \cong H_2(N(\hat{r}, \hat{s}, \hat{t}))$. As in the proof of the preceding lemma, $H_2(N(\hat{r}, \hat{s}, \hat{t}))$ is torsion-free; hence, $H_2(M^5) = H_2(N(\hat{r}, \hat{s}, \hat{t})) = 0$. Finally, $H_1(N(\hat{r}, \hat{s}, \hat{t}))$ is computed by means of the splitting sequence

$$0 \rightarrow \bigoplus_m \mathbb{Z} \rightarrow H_1(N(\hat{r}, \hat{s}, \hat{t})) \oplus \left(\bigoplus_m \mathbb{Z} \right) \rightarrow \bigoplus_m \mathbb{Z} \rightarrow 0. \quad \square$$

Proposition 7. *Let (Γ, γ) be a crystallization of a closed connected orientable 5-manifold M^5 . If there exists a cyclic permutation ε of Δ_5 , such that $\rho_\varepsilon(\Gamma) = \rho \leq 7$, then either $M^5 \cong \mathbb{S}^5$ or $M^5 \cong \#_m(\mathbb{S}^1 \times \mathbb{S}^4)$, $1 \leq m \leq \rho$.*

Proof. Set $\rho^{**} = \min\{\rho_{ij} | i, j \in \Delta_5, i \neq j\}$. By Proposition 3, the case $\rho^{**} = 0$ directly implies the thesis; hence, $\rho^{**} \geq 1$ may be supposed. Let us further set $\rho^* = \min\{\rho_i | i \in \Delta_5\}$. By Corollary 4, $\rho^* \geq 3$ may also be supposed. Now, formulae (7) and (8) imply that, if $\rho(\Gamma) \leq 7$, then $\rho^* \leq 4$ (since $\mathbf{g}_{\mathcal{B}} \geq 1$, for every $\mathcal{B} \subseteq \Delta_5$). If $r \in \Delta_5$ is such that $\rho_r = \rho^*$, then the only possible cases for formula (9) are:

- (a) $\rho_r = 3 = (1 + 1 + 1 + 1 + 2)/2$;
- (b) $\rho_r = 4 = (1 + 1 + 2 + 2 + 2)/2$;
- (c) $\rho_r = 4 = (1 + 1 + 1 + 2 + 3)/2$.

Note that the further case $\rho_r = 4 = (1 + 1 + 1 + 1 + 4)/2$ cannot hold by Remark C.

Anyway, in all the previous three cases (a), (b), (c), there exist s, t nonconsecutive in ε_r such that $\rho_r = \rho_{rs} + \rho_{rt}$; thus, Lemma 2 implies that $N(r, s, t)$ is homeomorphic to a 5-dimensional handlebody Y_m of genus m , with $\text{rk}(\pi_1(M^5)) \leq m \leq \rho$.

If $m = 0$, i.e., if $N(r, s, t) \cong \mathbb{D}^5$, then M^5 is simply connected and Lemma 5 concludes the proof. Let us assume $N(r, s, t) \cong Y_m$, $1 \leq m \leq \rho$. Set $\{u, v, w\} = \Delta_5 - \{r, s, t\}$; by the hypothesis on $\{r, s, t\}$, u may be supposed to be nonconsecutive both to v and to w (in ε). If $\rho_v = \rho_{uv} + \rho_{vw}$ (or $\rho_w = \rho_{uw} + \rho_{vw}$), then the thesis directly follows from Lemma 2 by making use of Corollary 4.4 and Theorem 2.1 of [CH].

Let us assume

$$(\$) \quad \rho_v - \rho_{uv} - \rho_{vw} \geq 1 \quad \text{and} \quad \rho_w - \rho_{uw} - \rho_{vw} \geq 1.$$

Formula (6) gives

$$\begin{aligned} \mathbf{g}_{uvw} &= \mathbf{g}_{uv} + \mathbf{g}_{vw} - 1 + (\rho_v - \rho_{uv} - \rho_{vw}) \geq \mathbf{g}_{uv} + \mathbf{g}_{vw}, \\ \mathbf{g}_{uvw} &= \mathbf{g}_{uw} + \mathbf{g}_{vw} - 1 + (\rho_w - \rho_{uw} - \rho_{vw}) \geq \mathbf{g}_{uw} + \mathbf{g}_{vw}; \end{aligned}$$

i.e.,

$$(+) \quad \mathbf{g}_{uw} \leq \mathbf{g}_{uvw} - \mathbf{g}_{vw} \quad \text{and} \quad \mathbf{g}_{uv} \leq \mathbf{g}_{uvw} - \mathbf{g}_{vw}.$$

By Lemma 6, $\chi(K(u, v, w)) = 1 - m \leq 0$; on the other hand,

$$\chi(K(u, v, w)) = 3 - (\mathbf{g}_{uw} + \mathbf{g}_{uv} + \mathbf{g}_{vw}) + \mathbf{g}_{uvw}.$$

Hence, $0 \leq \mathbf{g}_{uvw} - \mathbf{g}_{vw} \leq \mathbf{g}_{uv} + \mathbf{g}_{uw} - 3$, and, by (+), $\mathbf{g}_{uv} \geq 3$, $\mathbf{g}_{uw} \geq 3$.

Moreover, by making use of $\rho \leq 7$, $\rho^{**} \geq 1$, and relations (\$), we obtain from formula (5)

$$\begin{aligned} \mathbf{g}_{uv} &= 1 + \rho - \rho_u - (\rho_v - \rho_{uv}) \leq 3, \\ \mathbf{g}_{uw} &= 1 + \rho - \rho_u - (\rho_w - \rho_{uw}) \leq 3. \end{aligned}$$

Hence, $\mathbf{g}_{uv} = \mathbf{g}_{uw} = 3$ and $\mathbf{g}_{uvw} = \mathbf{g}_{vw} + 3$. This means that the two-dimensional complex $K(u, v, w)$ satisfies the hypotheses (i) and (ii) of Lemma 1b; moreover, hypothesis (iii) and (iv) hold for [G1, Proposition 3] and by Lemma 6 respectively. As a consequence, $K(u, v, w)$ collapses to a graph, and therefore $N(u, v, w) = N(\hat{r}, \hat{s}, \hat{t}) \cong Y_n$. Since $N(r, s, t) \cong Y_m$ is assumed, $m = n$ obviously holds; the thesis again follows from Corollary 4.4 and Theorem 2.1 of [CH]. $\square$

Proof of the main theorem and of Corollary I. The proof of the main theorem is a direct consequence of Proposition 7. On the other hand, the genus of $S^3 \times S^2$ is proved to be ≤ 8 in [GG]. Hence Corollary I again follows from Proposition 7. $\square$

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