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A GENERALIZATION OF THE ISOPERIMETRIC INEQUALITY ON S^2 AND FLAT TORI IN S^3

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ABSTRACT. A geometric inequality is proved for closed curves on S^2 which are regularly homotopic to simple closed curves. This generalizes the classical isoperimetric inequality for simple closed curves on S^2. The proof is based on the study of flat tori in S^3 and their images under the Gauss map.

INTRODUCTION

Let α be an oriented closed curve of class C^3 in the unit 2-sphere S^2. We denote the length of α by L, the oriented geodesic curvature by k, and the arclength parameter by s. In this paper we prove the following theorem.

Theorem. *Suppose that α is regularly homotopic to a simple closed curve. Then*

$$L^2 + \left(\int_\alpha k\, ds\right)^2 \geq 4\pi^2.$$

The equality holds if and only if α is a small circle.

If α is a simple closed curve, the Gauss-Bonnet theorem gives

$$\int_\alpha k = 2\pi - A,$$

where A is the area bounded by α. Thus our theorem implies the following classical isoperimetric inequality for simple closed curves in S^2:

$$L^2 + (2\pi - A)^2 \geq 4\pi^2.$$

By Smale's theorem [3], any regular closed curve in S^2 is regularly homotopic to either a circle traversed once or a circle traversed twice. For the latter case our theorem fails, since the quantity on the left-hand side in our inequality can be arbitrarily close to zero for small "figure-eight" curves.

Our theorem has come out of the study of flat surfaces in S^3. We now describe the correspondence between curves in S^2 and flat surfaces in S^3. Let γ be a tangent vector of α whose norm is $1/\sqrt{2}$. γ defines a closed curve in

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a round 2-sphere of radius $1/\sqrt{2}$. Let $G_{2,4}$ be the Grassmannian manifold of oriented 2-dimensional linear subspaces of $\mathbf{R}^4$. Equipped with the standard invariant Riemannian metric, $G_{2,4}$ is isometric to $S_1 \times S_2$, where each S_i ($i = 1, 2$) is a round 2-sphere of radius $1/\sqrt{2}$. We regard γ as a curve in S_1. Let Γ be a great circle in S_2. Using a method given in [3, 4], we can construct a flat torus T whose image under the Gauss map $G: T \to G_{2,4}$ is $\gamma \times \Gamma$. By a result of Weiner [4], if γ is regularly homotopic to a circle (or equivalently, if α is regularly homotopic to a circle), T has antipodal symmetry, i.e., $-x \in T$ for any $x \in T$. There exists a curve σ in T which connects x and $-x$ and is mapped by G onto γ with an element in S_2 fixed. The main ingredient of the proof is to express the distance between x and $-x$ in T in terms of the length and the geodesic curvature of α, via σ.

1. Gauss maps of flat tori in S^3

In this section we recall some results from [1, 2, 4] on flat tori in S^3 and their Gauss maps.

For an oriented surface M in $\mathbf{R}^4$ we define the Gauss map $G: M \to G_{2,4}$ by assigning each point x of M to the 2-dimensional linear subspace parallel to the tangent plane of M at x. $G_{2,4}$, equipped with the standard invariant metric, is isometric to $S_1 \times S_2$, where each S_i ($i = 1, 2$) is a round 2-sphere of radius $1/\sqrt{2}$.

Lemma 1 [1]. *Suppose that M is flat (i.e., the Gaussian curvature is identically zero) and the normal connection of M is flat. If the Gauss map G is regular on M, then $G(M)$ is locally the product of a curve γ_1 in S_1 and a curve γ_2 in S_2.*

Let T be a flat torus in S^3. Then the normal connection of T as a submanifold in $\mathbf{R}^4$ is flat and the Gauss map G is regular everywhere. Hence $G(T)$ is a finite covering of $\gamma_1 \times \gamma_2$, where γ_i ($i = 1, 2$) is a closed regular curve in S_i. Let σ_i be a curve in T which is mapped by G onto γ_i with an element in the other factor fixed.

Lemma 2. *For $i = 1, 2$ let K_i and κ_i be the oriented geodesic curvatures of σ_i and γ_i respectively. Then for any x in σ_i we have $K_i(x) = \sqrt{2}\kappa_i(G(x))$.*

Proof. Let N be a unit normal vector field of T in S^3 and $\{e_1, e_2\}$ be a set of orthonormal principal vectors of T in S^3. Let λ_i ($i = 1, 2$) be the principal curvature corresponding to e_i. Since T is flat, $\lambda_1\lambda_2 = -1$. We may assume that $\lambda_1 > 0$. If we set $\lambda_1 = \tan\beta$ and $\lambda_2 = -\cot\beta$, β defines a smooth function on T. Set

$$X_1 = \cos\beta e_1 - \sin\beta e_2, \qquad X_2 = \cos\beta e_1 + \sin\beta e_2,$$
$$e_3 = -\cos\beta x + \sin\beta N,. \qquad e_4 = \sin\beta x + \cos\beta N.$$

Since $G(x) = e_1(x) \wedge e_2(x)$, we have

$$\begin{aligned} dG(X_1) &= de_1(X_1) \wedge e_2 + e_1 \wedge de_2(X_1) \\ &= \langle de_1(X_1), e_3\rangle e_3 \wedge e_2 + \langle de_1(X_1), e_4\rangle e_4 \wedge e_2 \\ &\quad + \langle de_2(X_1), e_3\rangle e_1 \wedge e_3 + \langle de_2(X_1), e_4\rangle e_1 \wedge e_4 \\ &= e_1 \wedge e_4 - e_2 \wedge e_3. \end{aligned} \tag{1.1}$$

$\{e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4\}$ forms an orthonormal base of the tangent space of $G_{2,4} = S_1 \times S_2$. Since the tangent space of S_1 is spanned by $\{e_1 \wedge e_3 + e_2 \wedge e_4, e_1 \wedge e_4 - e_2 \wedge e_3\}$, (1.1) shows that $dG(X_1)$ is a tangent vector of γ_1. Thus X_1 is a unit tangent vector of σ_1, since $G(\sigma_1) = \gamma_1$. By a similar argument we see that X_2 is a unit tangent vector of σ_2. Let D denote the covariant differentiation on T. The oriented geodesic curvature K_1 of σ_1 is given by

$$\begin{aligned} K_1 &= \langle D_{X_1}X_1, -\sin\beta e_1 - \cos\beta e_2\rangle \\ &= X_1\beta - \sin\beta\langle D_{e_2}e_2, e_1\rangle - \cos\beta\langle D_{e_1}e_1, e_2\rangle. \end{aligned} \tag{1.2}$$

By the Codazzi equation, we have

$$\langle D_{e_1}e_1, e_2\rangle = \tan\beta e_2\beta, \qquad \langle D_{e_2}e_2, e_1\rangle = -\cot\beta e_1\beta. \tag{1.3}$$

It follows from (1.2) and (1.3) that

$$K_1 = 2X_1\beta. \tag{1.4}$$

By a similar computation, we have

$$K_2 = 2X_2\beta. \tag{1.5}$$

We denote by $\tilde{g}$ the standard invariant metric on $G_{2,4}$ and by $\tilde{D}$ the covariant differentiation corresponding to $\tilde{g}$. Since a unit tangent vector of γ_1 is given by $dG(X_1)/\sqrt{2} = (e_1 \wedge e_4 - e_2 \wedge e_3)/\sqrt{2}$,

(1.6)

$$\begin{aligned} \kappa_1 &= \tilde{g}\left(\tilde{D}_{\frac{dG(X_1)}{\sqrt{2}}}\frac{e_1\wedge e_4 - e_2\wedge e_3}{\sqrt{2}}, -\frac{e_1\wedge e_3 + e_2\wedge e_4}{\sqrt{2}}\right) \\ &= \frac{1}{2\sqrt{2}}\tilde{g}(de_1(X_1)\wedge e_4 + e_1\wedge de_4(X_1) - de_2(X_1)\wedge e_3 \\ &\qquad - e_2\wedge de_3(X_1), -e_1\wedge e_3 - e_2\wedge e_4) \\ &= \frac{1}{2\sqrt{2}}\tilde{g}(\langle D_{X_1}e_1, e_2\rangle e_2\wedge e_4 + \langle D_{X_1}e_4, e_3\rangle e_1\wedge e_3 \\ &\qquad - \langle D_{X_1}e_2, e_1\rangle e_1\wedge e_3 - \langle D_{X_1}e_3, e_4\rangle e_2\wedge e_4, -e_1\wedge e_3 - e_2\wedge e_4) \\ &= \frac{1}{2\sqrt{2}}(-\langle D_{X_1}e_1, e_2\rangle - \langle D_{X_1}e_4, e_3\rangle + \langle D_{X_1}e_2, e_1\rangle + \langle D_{X_1}e_3, e_4\rangle) \\ &= \frac{1}{2\sqrt{2}}(-\cos\beta\langle D_{e_1}e_1, e_2\rangle + \sin\beta\langle D_{e_2}e_1, e_2\rangle \\ &\qquad + \cos\beta\,\langle D_{e_1}e_3, e_4\rangle - \sin\beta\langle D_{e_2}e_3, e_4\rangle). \end{aligned}$$

Here we have

$$\langle D_{e_i}e_3, e_4\rangle = \langle D_{e_i}(-\cos\beta x + \sin\beta N), \sin\beta x + \cos\beta N\rangle = e_i\beta \tag{1.7}$$

for $i = 1, 2$. It follows from (1.3), (1.6), and (1.7) that

$$\kappa_1 = \sqrt{2}(\cos\beta e_1\beta - \sin\beta e_2\beta) = \sqrt{2}X_1\beta. \tag{1.8}$$

Similarly, we have

$$\kappa_2 = \sqrt{2} X_2 \beta. \tag{1.9}$$

Now the lemma follows from (1.4), (1.5), (1.8), and (1.9).

Let l_i be the length of γ_i for $i = 1, 2$. Since $G|_{\sigma_i} : \sigma_i \to \gamma_i$ is a finite covering and $|dG(X_i)| = \sqrt{2}$ by (1.1), σ_i is a closed curve in T whose length is an integral multiple of $l_i/\sqrt{2}$. We need the following lemma due to Weiner.

Lemma 3 [4]. *Suppose that either γ_1 or γ_2 is regularly homotopic to a simple closed curve. Then $G: T \to \gamma_1 \times \gamma_2$ is a double covering.*

We assume that γ_1 is regularly homotopic to a simple closed curve. Let u be the arclength parameter of σ_1. By Lemma 3, $G|_{\sigma_1} : \sigma_1 \to \gamma_1$ is a double covering and σ_1 is a closed curve in T whose length is $\sqrt{2}\, l_1$. Then

$$\sigma_1(u + \sqrt{2}\, l_1) = \sigma_1(u) \tag{1.10}$$

holds for any u. The following lemma is also a result of Weiner and a key to proving our theorem.

Lemma 4 [4]. *Suppose that γ_1 is regularly homotopic to a simple closed curve. Then $\sigma_1(u + l_1/\sqrt{2})$ is the antipodal point of $\sigma_1(u)$ in S^3.*

Remark. Lemma 4 implies the antipodal symmetry of certain flat tori in S^3; if either γ_1 or γ_2 is regularly homotopic to a simple closed curve, then $-x \in T$ for any $x \in T$.

2. Construction of a flat torus in S^3

Let $\alpha(s)$ be an oriented closed C^3 curve in S^2 which is parameterized by its arclength s. We set

$$\gamma(s) = \frac{1}{\sqrt{2}} \frac{d\alpha}{ds}.$$

$\gamma(s)$ defines an oriented closed C^2 curve in $S^2(1/\sqrt{2})$. Let k and κ be the oriented geodesic curvature of α and γ, respectively. By an easy computation, we have

$$\kappa = \sqrt{2}\,(1 + k^2)^{-3/2} \frac{dk}{ds}. \tag{2.1}$$

Since

$$\left| \frac{d\gamma}{ds} \right| = \sqrt{\frac{1 + k^2}{2}},$$

we have for any subarc $\gamma' = \gamma([s_1, s_2])$

$$\begin{aligned} \int_{\gamma'} \kappa &= \int_{s_1}^{s_2} \sqrt{2}(1 + k^2)^{-3/2} \frac{dk}{ds} \sqrt{\frac{1 + k^2}{2}}\, ds \\ &= \int_{s_1}^{s_2} \frac{1}{1 + k^2} \frac{dk}{ds}\, ds = \arctan k(s_2) - \arctan k(s_1). \end{aligned} \tag{2.2}$$

In particular, we have

$$\int_{\gamma} \kappa = 0, \tag{2.3}$$

and, for any subarc γ' of γ,

$$\left|\int_{\gamma'} \kappa\right| < \pi. \tag{2.4}$$

We regard γ as a curve in the first factor S_1 of $G_{2,4} = S_1 \times S_2$. Let Γ be a great circle in S_2. Using a method that is first given by the author [2] and then developed by Weiner [4], one can construct a flat torus T_α immersed in S^3 whose image under the Gauss map $G: T_\alpha \to G_{2,4}$ is a finite covering of $\gamma \times \Gamma$. We note that the conditions (2.3) and (2.4) imply that T_α is regular at every point.

3. Generalized isoperimetric inequality on S^2

Let α be an oriented closed C^3 curve in S^2, and let T_α be a flat torus in S^3 which is constructed from α in §2. Fix a point x_0 in T_α. Then there exists a closed curve σ in T_α passing through x_0 such that the Gauss map G maps σ onto γ with an element in S_2 fixed. Let $\pi: \tilde{T}_\alpha \to T_\alpha$ be the universal covering of T_α. Let $\tilde{x}_0$ be a point in $\tilde{T}_\alpha$ such that $\pi(\tilde{x}_0) = x_0$, and let $\tilde{\sigma}$ be a curve in $\tilde{T}_\alpha$ passing through $\tilde{x}_0$ such that $\pi(\tilde{\sigma}) = \sigma$.

Theorem. *Let α be an oriented closed C^3 curve in S^2. Suppose that α is regularly homotopic to a simple closed curve in S^2. Then*

$$L^2 + \left(\int_\alpha k\, ds\right)^2 \geq 4\pi^2,$$

where L is the length of α, k is the oriented geodesic curvature of α, and s is the arclength parameter of α. The equality holds if and only if α is a small circle.

Proof. Let (x, y) be the Cartesian coordinate system on $\tilde{T}_\alpha = \mathbf{R}^2$. We write

$$\tilde{\sigma}(u) = (x(u), y(u)),$$

where u is the arclength parameter of $\tilde{\sigma}$. We may assume that $\tilde{x}_0 = (0, 0)$ and

$$\tilde{\sigma}(0) = (0, 0), \qquad \frac{d\tilde{\sigma}}{du}(0) = (1, 0).$$

Let l be the length of γ. If α is regularly homotopic to a simple closed curve in S^2, then so is γ. By Lemma 4, $\sigma(l/\sqrt{2})$ is the antipodal point of $\sigma(0)$ and the distance between $\tilde{\sigma}(0)$ and $\tilde{\sigma}(l/\sqrt{2})$ in $\tilde{T}_\alpha$ is not less than π. This gives

$$\left|\tilde{\sigma}\left(\frac{l}{\sqrt{2}}\right)\right|^2 \geq \pi^2. \tag{3.1}$$

If we write

$$\frac{d\tilde{\sigma}}{du} = (\cos\theta(u), \sin\theta(u)), \tag{3.2}$$

$\theta(u)$ is given by

$$\theta(u) = \int_0^u K(u)\, du, \tag{3.3}$$

where K is the oriented geodesic curvature of $\tilde{\sigma}$. Let t be the arclength parameter of γ and κ be the oriented geodesic curvature of γ. By (1.1), we have

$$\frac{dt}{du} = \sqrt{2}. \tag{3.4}$$

Combining (2.2), (3.3), (3.4), and Lemma 2, we obtain

$$\theta(u) = \int_0^t \sqrt{2}\kappa(t) \cdot \frac{1}{\sqrt{2}}\, dt = \arctan k(s) - \arctan k(0). \tag{3.5}$$

It follows from (3.2) and (3.5) that

$$\frac{d\tilde{\sigma}}{du} = \left(\frac{1 + k(s)k(0)}{\sqrt{1+k(s)^2}\sqrt{1+k(0)^2}}, \frac{k(s) - k(0)}{\sqrt{1+k(s)^2}\sqrt{1+k(0)^2}} \right). \tag{3.6}$$

Thus we have

$$\begin{aligned} \tilde{\sigma}\left(\frac{l}{\sqrt{2}}\right) &= \int_0^{l/\sqrt{2}} \frac{d\tilde{\sigma}}{du}\, du = \int_0^L \frac{d\tilde{\sigma}}{du}\frac{du}{dt}\frac{dt}{ds}\, ds \\ &= \int_0^L \left(\frac{1 + k(s)k(0)}{2\sqrt{1+k(0)^2}}, \frac{k(s) - k(0)}{2\sqrt{1+k(0)^2}} \right) ds \\ &= \frac{1}{2\sqrt{1+k(0)^2}} \left(L + k(0)\int_0^L k\, ds,\ -k(0)L + \int_0^L k\, ds \right). \end{aligned} \tag{3.7}$$

It follows from (3.1) and (3.7) that

$$\begin{aligned} \pi^2 &\le \frac{1}{4(1+k(0)^2)} \left(\left(L + k(0)\int_\alpha k\, ds \right)^2 + \left(-k(0)L + \int_\alpha k\, ds \right)^2 \right) \\ &= \frac{1}{4}\left(L^2 + \left(\int_\alpha k\, ds \right)^2 \right) \quad . \end{aligned} \tag{3.8}$$

This proves the inequality of the theorem.

Suppose that the equality holds. Then $\tilde{\sigma}([0, l/\sqrt{2}])$ must be a geodesic segment in T_α. Thus the oriented geodesic curvature K of σ vanishes for $0 \le s \le l/\sqrt{2}$ and, by Lemma 2, the geodesic curvature κ of γ vanishes. Then (2.1) implies that $k(s)$ is constant, and hence α is a small circle.

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