PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 2, February 1994

STRONG MOMENT PROBLEMS FOR RAPIDLY DECREASING SMOOTH FUNCTIONS

ANA LÍA DURÁN AND RICARDO ESTRADA

(Communicated by J. Marshall Ash)

ABSTRACT. It is shown that the existence of rapidly decreasing smooth solutions of various moment problems follows from the theorem of Ritt on the existence of analytic functions with a prescribed asymptotic power series at the vertex of a given sector.

1. INTRODUCTION

The problem of moments, as well as its many generalizations, have been the object of research for over a century. In its basic form it consists of finding a positive function $\phi(x)$, defined on a given interval I, that satisfies

$$\int_I x^n \phi(x)\, dx = \mu_n, \qquad n \in \mathbb{N}, \tag{1.1}$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ is a given sequence of real or complex numbers. We refer to [1, 17] for an account of the classical aspects of this problem.

The moments play a rather important role in several areas of current interest, including the theory of asymptotic expansions of generalized functions [7, 8, 10], the theory of orthogonal polynomials [12, 14], and the theory of distributional solutions of differential and functional equations [13, 19].

Recently, Durán [6] proved that when $I = [0, \infty)$, for an arbitrary sequence $\{\mu_n\}_{n\in\mathbb{N}}$, problem (1.1) admits solutions of the class $\mathscr{S}(0, \infty) = \{\phi \in \mathscr{S}: \phi(x) = 0, x \leq 0\}$. Here $\mathscr{S} = \mathscr{S}(\mathbb{R})$ is the Schwartz space of rapidly decreasing smooth functions [16]; namely,

$$\mathscr{S}(\mathbb{R}) = \left\{\phi \in C^\infty(\mathbb{R}): \lim_{|x|\to\infty} x^n \phi^{(m)}(x) = 0, n, m \in \mathbb{N}\right\}.$$

The arbitrariness of the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ had already been obtained by Boas [4], who worked in the class of functions of bounded variation. This contrasts with the situation when I is a compact interval, since growth conditions on certain linear combinations of the μ_n's are required for existence, even in the case when ϕ is a distribution [9].

Received by the editors October 31, 1991 and, in revised form, May 27, 1992.
1991 *Mathematics Subject Classification*. Primary 44A60.

In the present article we show that this result as well as the existence of solutions of more general problems such as the *strong moment problem*,

$$\int_{-\infty}^{\infty} x^n \phi(x)\,dx = \mu_n, \qquad n \in \mathbb{Z}, \tag{1.2}$$

can be obtained from the following theorem of Ritt [18, p. 41]:

Theorem. *Let* $\{b_n\}_{n\in\mathbb{N}}$ *be a sequence of complex numbers. Then there exists a bounded analytic function* $F(z)$ *defined in the sector* $S_a(\alpha, \beta) = \{re^{i\theta} : 0 < r < a, \alpha < \theta < \beta\}$ *with the asymptotic power series*

$$F(z) \sim \sum_{n=0}^{\infty} b_n z^n, \qquad z \to 0,\ z \in S_a(\alpha, \beta). \quad \square \tag{1.3}$$

Actually, the function $F(z)$ can be chosen analytic and bounded in the whole sector $S(\alpha, \beta) = S_\infty(\alpha, \beta) = \{re^{i\theta} : r > 0, \alpha < \theta < \beta\}$. This can be obtained from the result for $S_a(\alpha, \beta)$, $a < \infty$, as follows. Write $F(z) = f(h(z))$, where $h: S(\alpha, \beta) \to S_a(\alpha, \beta)$ is a conformal map with $h(0) = 0$ and $h(z) \sim z + c_2 z^2 + c_3 z^3 + \cdots$, as $z \to 0$, and where $f(\omega)$ is chosen as in the theorem with asymptotic development $f(\omega) \sim d_0 + d_1\omega + d_2\omega^2 + \cdots$, as $\omega \to 0$, with the d_i's chosen in such a way that, formally,

$$\sum_{j=0}^{\infty} d_j \left(z + \sum_{n=2}^{\infty} c_n z^n \right)^j = \sum_{n=0}^{\infty} b_n z^n.$$

2. Solution of the moment problem

In this section we use the theorem of Ritt to give a proof of the existence of solutions in the class $\mathcal{S}(0, \infty)$ of the problem

$$\int_0^{\infty} x^n \phi(x)\,dx = \mu_n, \qquad n \in \mathbb{N}, \tag{2.1}$$

where $\{\mu_n\}_{n\in\mathbb{N}}$ is an arbitrary sequence. This result was obtained by Durán [6], who gave a different proof.

We first recall [2] that a function $\psi \in \mathcal{S}(\mathbb{R})$ is the Fourier transform

$$\psi(u) = \hat{\phi}(u) = \int_0^{\infty} e^{ixu} \phi(x)\,dx \tag{2.2}$$

of a function ϕ of the class $\mathcal{S}(0, \infty)$ if and only if it can be extended to a bounded continuous function $\Psi(z)$ in the upper half plane $\operatorname{Im} z \geq 0$, analytic in $\operatorname{Im} z > 0$ and vanishing as $z \to \infty$.

Since

$$\frac{d^n \hat{\phi}(0)}{du^n} = i^n \int_0^{\infty} x^n \phi(x)\,dx, \tag{2.3}$$

problem (2.1) is equivalent to that of finding $\psi = \hat{\phi}$ in the class $\mathcal{F}(\mathcal{S}(0, \infty))$ that satisfies

$$\psi^{(n)}(0) = i^n \mu_n, \qquad n \in \mathbb{N}. \tag{2.4}$$

We construct ψ as follows. Let

$$G(z) = e^{(1-i)(z+i)^{1/2}}, \tag{2.5}$$

where the branch of the square root is chosen so as to assure that $G(z) \to 0$ as $z \to \infty$ in the upper half plane $\operatorname{Im} z \geq 0$. Let the sequence $a_0, a_1, a_2, \ldots$ be defined as

$$\frac{1}{G(z)} = a_0 + a_1 z + a_2 z^2 + \cdots, \qquad |z| < 1. \tag{2.6}$$

Let $\delta \in (0, \pi/2)$, and let $F(z)$ be a bounded analytic function in the sector S: $-\delta < \operatorname{Arg} z < \pi + \delta$ with the asymptotic power series

$$F(z) \sim b_0 + b_1 z + b_2 z^2 + \cdots, \qquad z \to 0,\ z \in S, \tag{2.7}$$

where

$$b_n = \sum_{k=0}^{n} \frac{i^k}{k!} \mu_k a_{n-k}. \tag{2.8}$$

Let $\Psi(z) = F(z)G(z)$. Then Ψ is analytic in the sector S, and $\Psi(z) = o(z^{-n})$ for each $n \in \mathbb{N}$ as $z \to \infty$ within S. Also,

$$\Psi(z) \sim \sum_{n=0}^{\infty} \frac{i^n \mu_n z^n}{n!}, \qquad z \to 0,\ z \in S. \tag{2.9}$$

It follows that, if ψ is the restriction of Ψ to the real axis, then $\psi \in \mathscr{F}(\mathscr{S}(0, \infty))$ and ψ satisfies (2.4), as required.

3. The strong moment problem

In this section we consider the following strong moment problem: Find $\phi \in \mathscr{S}(\mathbb{R})$ that satisfies $\phi^{(n)}(0) = 0$, $n \in \mathbb{N}$, and

$$\int_{-\infty}^{\infty} x^n \phi(x)\, dx = \mu_n, \qquad n \in \mathbb{Z}, \tag{3.1}$$

where $\{\mu_n\}_{n \in \mathbb{Z}}$ is an arbitrary sequence. We refer to [3, 11, 15] for the study of this problem in the class of the positive Radon measures.

Let $\psi(u) = \hat{\phi}(u)$. Then $\psi \in \mathscr{S}(\mathbb{R})$, and (3.1) is equivalent to the two sets of conditions

$$\psi^{(n)}(0) = i^n \mu_n, \qquad n = 0, 1, 2, \ldots, \tag{3.2a}$$

$$\int_0^{\infty} \mu^n \psi(\mu)\, d\mu = n!\, i^{n+1} \mu_{-n-1}, \qquad n = 0, 1, 2, \ldots. \tag{3.2b}$$

Using the construction of §2, we can find $\psi_1 \in \mathscr{S}(\mathbb{R})$ such that $\psi_1^{(n)}(0) = i^n \mu_n$, $n = 0, 1, 2, \ldots$. We can also find $\psi_2 \in \mathscr{S}(0, \infty)$ such that

$$\int_0^{\infty} \mu^n \psi_2(\mu)\, d\mu = n!\, i^{n+1} \mu_{-n-1} - \int_0^{\infty} \mu^n \psi_1(\mu)\, d\mu. \tag{3.3}$$

Then $\psi = \psi_1 + \psi_2$ belongs to $\mathscr{S}(\mathbb{R})$ and satisfies (3.2a,b). However, if $\hat{\phi} = \psi$, the condition $\phi^{(n)}(0) = 0$, $n \in \mathbb{N}$, might not be satisfied. Following a suggestion of Professor C. Berg, we solve this by choosing $\psi_3 \in \mathscr{S}(-\infty, 0)$ such that

$$\int_{-\infty}^{\infty} y^n \psi_3(y)\, dy = -n!\, i^{n+1} \mu_{-n-1} - \int_{-\infty}^{0} y^n \psi_1(y)\, dy, \tag{3.4}$$

for $n \in \mathbb{N}$, and define $\psi = \psi_1 + \psi_2 + \psi_3$. This function ψ satisfies (3.2a, b) but also

$$\int_{-\infty}^{\infty} y^n \psi(y)\, dy = 0, \qquad n \in \mathbb{N}, \tag{3.5}$$

and thus if $\hat{\phi} = \psi$, then $\phi^{(n)}(0) = 0$, $n \in \mathbb{N}$.

Actually, the strong moment problem can be solved in $\mathscr{S}(0, \infty)$; that is, there are $\sigma \in \mathscr{S}(0, \infty)$ with

$$\int_0^{\infty} x^n \sigma(x)\, dx = \mu_n, \qquad n \in \mathbb{Z}. \tag{3.6}$$

To see it, let $\{\mu_n\}_{n\in\mathbb{Z}}$ be any sequence, and let $\phi \in \mathscr{S}(\mathbb{R})$ with $\phi^{(n)}(0) = 0$, $n \in \mathbb{N}$, be a solution of the strong problem

$$\int_{-\infty}^{\infty} x^n \phi(x)\, dx = \begin{cases} \mu_{n/2}, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases} \tag{3.7}$$

Let $\rho \in \mathscr{S}(0, \infty)$ be given by $\rho(x) = \phi(x) + \phi(-x)$, $x \geq 0$. Then

$$\int_0^{\infty} u^{2n} \rho(u)\, du = \mu_n, \qquad n \in \mathbb{Z}. \tag{3.8}$$

Set

$$\sigma(x) = \frac{\rho(x^{1/2})}{2x^{1/2}}, \qquad x \geq 0. \tag{3.9}$$

Then $\sigma \in \mathscr{S}(0, \infty)$, and it satisfies (3.6).

Summarizing, we have shown the following result:

Theorem. *Let $\{\mu_n\}_{n\in\mathbb{Z}}$ be an arbitrary sequence. Then there exist functions $\phi \in \mathscr{S}(0, \infty)$ that satisfy*

$$\int_0^{\infty} x^n \phi(x)\, dx = \mu_n \tag{3.10}$$

for each $n \in \mathbb{Z}$.

4. Further remarks

Our results imply the existence of smooth functions $\phi_k \in \mathscr{S}(0, \infty)$, $k = 0, 1, 2, \ldots$, that satisfy the moment problem

$$\int_0^{\infty} x^n \phi_k(x)\, dx = \delta_{nk}, \tag{4.1}$$

where $\delta_{nk} = 1$ if $n = k$, $\delta_{nk} = 0$ if $n \neq k$ is the Kronecker delta. The functions $\phi_k(x)$ can be used as "smooth versions" of the Dirac delta functions $(-1)^k \delta^{(k)}(x)/k!$, which also satisfy

$$\left\langle \frac{(-1)^k \delta^{(k)}(x)}{k!}, x^n \right\rangle = \delta_{nk}. \tag{4.2}$$

In particular, if $\psi \in \mathscr{S}(\mathbb{R})$ has moments $\mu_n = \langle \psi(x), x^n \rangle$, $n = 0, 1, 2, \ldots$, then we could expect some relationship between $\psi(x)$ and the series

$\sum_{n=0}^{\infty} \mu_n \phi_n(x)$. Although the series is not necessarily equal to ψ (it could even be divergent), we do have the asymptotic relation

$$\psi(\lambda x) \sim \mu_0 \phi_0(\lambda x) + \mu_1 \phi_1(\lambda x) + \mu_2 \phi_2(\lambda x) + \cdots, \quad \text{as } \lambda \to \infty \tag{4.3}$$

in the space $\mathscr{S}'(\mathbb{R})$. The interpretation of (4.3) is in the distributional sense; namely, that for each $\rho \in \mathscr{S}(\mathbb{R})$ we have

$$\int_{-\infty}^{\infty} \psi(\lambda x)\rho(x)\,dx = \sum_{j=0}^{n} \mu_j \int_{-\infty}^{\infty} \phi_j(\lambda x)\rho(x)\,dx + O(\lambda^{-n-2}), \quad \text{as } \lambda \to \infty. \tag{4.4}$$

Relation (4.3) can be interpreted as saying that $\{\phi_k\}_{k\in\mathbb{N}}$ is an "asymptotic basis" of the space $\mathscr{S}(\mathbb{R})$. Actually, (4.3) is a smooth version of the *moment asymptotic expansion* [10]

$$f(\lambda x) \sim \sum_{n=0}^{\infty} \frac{(-1)^n \mu_n \delta^{(n)}(x)}{n!\lambda^{n+1}}, \quad \text{as } \lambda \to \infty, \tag{4.5}$$

valid if f is a generalized function of distributional rapid decay at infinity, where the $\mu_n = \langle f(x), x^n \rangle$ are the moments.

We would like to remark that our results imply that the class $\mathscr{M}$ of entire functions of the form

$$\Phi(z) = \int_0^{\infty} x^z \phi(x)\,dx, \tag{4.6}$$

where $\phi \in \mathscr{S}(0, \infty)$, satisfies the following interpolation property: for each sequence $\{\mu_n\}_{n\in\mathbb{Z}}$ there are $\Phi \in \mathscr{M}$ that satisfy

$$\Phi(n) = \mu_n, \qquad n \in \mathbb{Z}. \tag{4.7}$$

The class $\mathscr{M}$ is larger than the class of entire functions of exponential type [5]. Indeed, if $\Phi \in \mathscr{M}$ then there exists $\sigma \in C^{\infty}(\mathbb{R})$ such that $\sigma(t) = o(e^{-n|t|})$ as $|t| \to \infty$ for each $n \in \mathbb{N}$, with

$$\Phi(z) = \int_{-\infty}^{\infty} e^{zt}\sigma(t)\,dt. \tag{4.8}$$

For functions of exponential type, the $\sigma(t)$ has compact support. Actually, there are sequences $\{\mu_n\}_{n\in\mathbb{Z}}$ for which (4.7) does not have solutions of exponential type.

Acknowledgments

We thank Professor Christian Berg for pointing out a mistake in an earlier draft of the paper and for suggesting the use of (3.4) in order to solve it.

References

1. N. I. Akhiezer, *The classical moment problem*, Oliver and Boyd, Edinburgh, 1965.
2. E. J. Beltrami and M. R. Wohlers, *Distributions and the boundary values of analytic functions*, Academic Press, New York, 1966.
3. C. Berg, J. P. R. Christensen, and P. Ressel, *Harmonic analysis on semigroups*, Springer-Verlag, New York, 1984.
4. R. P. Boas, *The Stieljes moment problem for functions of bounded variation*, Bull. Amer. Math. Soc. **45** (1939), 399–404.

5. ———, *Entire functions*, Academic Press, New York, 1954.

6. A. J. Durán, *The Stieljes moment problem for rapidly decreasing functions*, Proc. Amer. Math. Soc. **107** (1989), 731–741.

7. R. Estrada, *The asymptotic expansion of certain series considered by Ramanujan*, Appl. Anal. **43** (1992), 191–228.

8. R. Estrada, J. Gracia-Bondía, and J. Várilly, *On asymptotic expansions of twisted products*, J. Math. Phys. **30** (1989), 2789–2796.

9. R. Estrada and R. P. Kanwal, *Moment sequences for a class of distributions*, Complex Variables Theory Appl. **9** (1987), 31–39.

10. ———, *A distributional theory for asymptotic expansions*, Proc. Roy. Soc. London Ser. A **428** (1990), 399–430.

11. W. B. Jones, W. J. Thron, and O. Njåstad, *Orthogonal Laurent polynomials and the strong Hamburger moment problem*, J. Math. Anal. Appl. **98** (1984), 528–554.

12. A. M. Krall, R. P. Kanwal, and L. L. Littlejohn, *Distributional solutions of ordinary differential equations*, Canad. Math. Soc. Conf. Proc. **8** (1987), 227–246.

13. L. L. Littlejohn and R. P. Kanwal, *Distributional solutions of the hypergeometric differential equation*, J. Math. Anal. Appl. **122** (1987), 325–345.

14. R. D. Morton and A. M. Krall, *Distributional weight functions for orthogonal polynomials*, SIAM J. Math. Anal. **9** (1978), 604–626.

15. O. Njåstad, *An extended Hamburger moment problem*, Proc. Edinburgh Math. Soc. **28** (1985), 167–183.

16. L. Schwartz, *Théorie des distributiones*, Hermann, Paris, 1957.

17. J. A. Shohat and J. D. Tamarkin, *The problem of moments*, Amer. Math. Soc., Providence, RI, 1943.

18. W. Wasow, *Asymptotic expansions for ordinary differential equations*, Wiley, New York, 1965.

19. J. Wiener, *Generalized function solutions of differential and functional differential equations*, J. Math. Anal. Appl. **88** (1982), 170–182.

Escuela de Matemática, Universidad de Costa Rica, San José, Costa Rica
E-mail address, R. Estrada: RESTRADA@UCRVM2