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MODULE CATEGORIES WITHOUT SHORT CYCLES ARE OF FINITE TYPE

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ABSTRACT. Let A be an artin algebra. An indecomposable finitely generated A-module X is said to be on a short cycle if there exists an indecomposable finitely generated A-module Y and two nonzero noninvertible maps $f: X \to Y$ and $g: Y \to X$. If there are no short cycles we show that there exist only finitely many indecomposable A-modules up to isomorphism.

Throughout this paper A will denote an artin algebra over a commutative artin ring R, $\operatorname{mod} A$ the category of finitely generated left A-modules, and $\mathfrak{R}$ the Jacobson radical of $\operatorname{mod} A$. The artin algebra A is said to be of *finite representation type* if there exist only finitely many indecomposable modules in $\operatorname{mod} A$ up to isomorphism. We do not distinguish between an indecomposable module X in $\operatorname{mod} A$ and its isomorphism class. A *path* in $\operatorname{mod} A$ is a sequence $(X_0, X_1, \dots, X_s)$ of indecomposable modules in $\operatorname{mod} A$ such that $\mathfrak{R}(X_{i-1}, X_i) \neq 0$ for all $1 \leq i \leq s$. If $s \geq 1$ and $X_0 = X_s$, then the path $(X_0, X_1, \dots, X_s)$ is called a *cycle* in $\operatorname{mod} A$. If $s = 2$ and $X_0 = X_2$, then the path (X_0, X_1, X_2) is called a *short cycle* in *modA*. An indecomposable module X in $\operatorname{mod} A$ is said to be *directing* if it does not occur in any cycle in $\operatorname{mod} A$.

The aim of this note is to show that if A is an artin algebra such that $\operatorname{mod} A$ contains no short cycle, then A is of finite representation type.

This generalizes the following result by Ringel [8, (2.4.9′)]. A finite-dimensional algebra A over an algebraically closed field is of finite representation type if every indecomposable module in $\operatorname{mod} A$ is directing.

We point out that the proof of our result is obtained from a combination of Ringel's methods with methods and results of [3] and [7].

Note that the class of artin algebras whose module categories contain no short cycle is substantially larger than that of artin algebras whose module categories contain no cycle (see [3]).

We keep the notation introduced as before. The composition of two maps $f: X \to Y$ and $g: Y \to Z$ is denoted by fg; so we write maps in $\operatorname{mod} A$ on the right. We denote by Γ_A the Auslander-Reiten quiver of A and by τ the Auslander-Reiten translation DTr. Recall from [3] that an indecomposable

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module X in $\operatorname{mod} A$ is said to be *in the middle of a short chain* if there exists a chain $Y \to X \to \tau Y$ of nonzero maps in $\operatorname{mod} A$ where Y is a nonprojective indecomposable module. The following observation is useful.

Lemma 1. *Let A be an artin algebra, and let X be an indecomposable module in* $\operatorname{mod} A$. *Then the following are equivalent:*

(a) *X is on a short cycle in* $\operatorname{mod} A$.
(b) *X is in the middle of a short chain in* $\operatorname{mod} A$.

Proof. It is shown in [7, (1.6)] that (b) implies (a). For the converse, assume that $\operatorname{mod} A$ contains a short cycle (X, Y, X), say, with nonzero nonisomorphisms $f: X \to Y$ and $g: Y \to X$. If $fg = 0$, then there exists by [5, §1] a nonprojective indecomposable module W with $\operatorname{Hom}(W, X) \neq 0$ and $\operatorname{Hom}(X, \tau W) \neq 0$. So X is in the middle of a short chain. If $fg \neq 0$, then let $t > 1$ be the minimal positive integer such that $(fg)^t = 0$. Let $f' = fg$ and $g' = (fg)^{t-1}$. Then $f'g' = 0$. Applying the result in [5] again, we get that X is in the middle of a short chain.

In the next lemma we will derive some homological properties of module categories without short cycles.

Lemma 2. *Let A be an artin algebra such that* $\operatorname{mod} A$ *contains no short cycle. Let X be an indecomposable module in* $\operatorname{mod} A$. *Then* $\operatorname{End}(X)$ *is a division ring, and* $\operatorname{Ext}^1_A(X, X) = \operatorname{Ext}^2_A(X, X) = 0$.

Proof. It is easy to see that $\operatorname{End}(X)$ is a division ring, since X is not on a short cycle. Moreover, $\operatorname{Hom}(X, \tau X) = 0$, since X is not in the middle of a short chain by Lemma 1. Being an epimorphic image of $\operatorname{Hom}(X, \tau X)$, $\operatorname{Ext}^1_A(X, X) = 0$. For the last assertion, let $0 \to X \to I \to X' \to 0$ be an exact sequence in $\operatorname{mod} A$, with I the injective envelope of X. Then $\operatorname{Ext}^2_A(X, X) \cong \operatorname{Ext}^1_A(X, X')$. Let I' be an indecomposable summand of I. Then $\operatorname{Hom}(X, I') \neq 0$. Since I' is not in the middle of a short chain, we see that $\operatorname{Hom}(I', \tau X) = 0$; hence, $\operatorname{Hom}(X', \tau X) = 0$. Therefore, $\operatorname{Ext}^1_A(X, X') = 0$, since it is an epimorphic image of $\operatorname{Hom}(X', \tau X)$.

Let ${}_s\Gamma_A$ be the full subquiver of Γ_A, obtained by deleting all modules whose τ-orbits contain either projective modules of injective modules and all arrows attached to these modules. Then the connected components of the quiver ${}_s\Gamma_A$ are called *stable components of* Γ_A. A stable component Γ of Γ_A is said to be *τ-periodic* if Γ contains only τ-periodic modules.

Lemma 3. *Let A be an artin algebra. If Γ_A has infinitely many τ-orbits, then Γ_A has at least one infinite stable component.*

Proof. Assume that Γ_A contains infinitely many τ-orbits. Let Γ be a finite connected component of ${}_s\Gamma_A$. Then Γ is not a connected component of Γ_A, since A is of infinite representation type [1]. By definition of ${}_s\Gamma_A$ we infer that Γ intersects with the τ-orbit of a module which is an immediate predecessor of some projective or injective module. Since Γ_A is locally finite and contains only finitely many projective and finitely many injective modules, we conclude that the union of all finite connected components of ${}_s\Gamma_A$ contains only finitely many τ-orbits of Γ_A. Hence Γ_A has at least one infinite stable component, since Γ_A contains infinitely many τ-orbits.

Let $f\colon X \to Y$ be an irreducible map in $\operatorname{mod} A$. Recall from [6] that the *left degree* $d_l(f)$ of f is infinite if, for each integer $n \geq 0$ and each map $g \in \mathfrak{R}^n \setminus \mathfrak{R}^{n+1}$, we have that $gf \notin \mathfrak{R}^{n+2}$; otherwise, $d_l(f)$ is the least integer m such that there is a map g in $\mathfrak{R}^m \setminus \mathfrak{R}^{m+1}$ such that $gf \in \mathfrak{R}^{m+2}$.

Let

$$X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n$$

be a chain of irreducible maps in $\operatorname{mod} A$. It follows easily from the definition that if $d_l(f_i) \geq n$ for all $1 \leq i \leq n$, then the composition $f_1 \cdots f_n$ is in $\mathfrak{R}^n \setminus \mathfrak{R}^{n+1}$.

Lemma 4. *Let A be an artin algebra. Assume that each τ-orbit of Γ_A contains only finitely many modules. Then either A is of finite representation type, or* $\operatorname{mod} A$ *contains short cycles.*

Proof. Assume that A is of infinite representation type. Then it follows from our assumption that Γ_A has infinitely many τ-orbits. By Lemma 3 there exists an infinite stable component Γ of Γ_A. It is clear that Γ contains only τ-periodic modules. Then it follows from [4] that Γ is a stable tube, say, of rank r. Let X be a module in Γ with quasi-length $2r$, and let $Y \to X$ be the arrow pointing to the mouth. Choose irreducible maps $f_i\colon \tau^i Y \to \tau^i X$ and $g_i\colon \tau^{i+1} X \to \tau^i Y$ for $0 \leq i \leq r-1$. Since X has quasi-length $2r$, there exists a sectional path

$$Z_1 \to Z_2 \to \cdots \to Z_{2r-1} \to Z_{2r} = X$$

in Γ such that $Z_{2r-1} \oplus Y$ is a summand of the middle term of the almost split sequence ending with X. Hence by [6, (1.6)] the left degree of f_0 is greater than or equal to $2r$. The same argument shows that all irreducible maps f_i and g_i with $0 \leq i \leq r-1$ have left degree greater than or equal to $2r$. Hence the composition $g_{r-1} f_{r-1} \cdots g_0 f_0$ is in $\mathfrak{R}^{2r} \setminus \mathfrak{R}^{2r+1}$. Thus we get a short cycle in $\operatorname{mod} A$.

Recall that a module X in $\operatorname{mod} A$ is *sincere* if every simple module in $\operatorname{mod} A$ occurs as a composition factor of X.

Theorem. *Let A be an artin algebra such that* $\operatorname{mod} A$ *contains no short cycle. Then A is of finite representation type.*

Proof. As in [8, (2.4.9′)] we may assume that there exists a sincere indecomposable module X in $\operatorname{mod} A$. Then it follows from [7, (3.5)] that the global dimension of A is at most two. Let q_A be the homological quadratic form on the Grothendieck group $K_0(A)$ of A. For a module Z in $\operatorname{mod} A$ we denote by [Z] the corresponding element in $K_0(A)$, and for an R-module M we denote by $l_R(M)$ the length of M over R. Then

$$q_A([Z]) = l_R(\operatorname{Hom}(Z, Z)) - l_R(\operatorname{Ext}^1_A(Z, Z)) + l_R(\operatorname{Ext}^2_A(Z, Z)).$$

Note that q_A is a quadratic form with integral coefficients. Assume that $K_0(A)$ is of rank n. We identify $K_0(A)$ with $\mathbf{Z}^n$.

An element $0 \neq x = (x_1, \ldots, x_n) \in \mathbf{Z}^n$ with $x_i \geq 0$ is called positive. A quadratic form q on $\mathbf{Z}^n$ is called weakly positive if $q(x) > 0$ for all positive x.

We claim that q_A is weakly positive. Indeed the usual argument applies. Let $0 \neq x = (x_1, \ldots, x_n) \in \mathbf{Z}^n$ with $x_i \geq 0$. Choose a module Z in $\operatorname{mod} A$ such that $[Z] = x$ and $l_R(\operatorname{End}_A(Z))$ is minimal among all those modules Y in $\operatorname{mod} A$ with $[Y] = x$. Assume that $Z = \bigoplus_1^r Z_i$ with the Z_i indecomposable. Then by Lemma 2 we have that $\operatorname{Ext}_A^1(Z_i, Z_i) = 0$ for all $1 \leq i \leq r$, and by [8, Lemma 1. (2.3)] we have that $\operatorname{Ext}_A^1(Z_i, Z_j) = 0$ for all $i \neq j$. Thus

$$q_A(x) = q_A([Z]) = l_r(\operatorname{End}(Z)) + l_R(\operatorname{Ext}_A^2(Z, Z)) > 0,$$

since $Z \neq 0$.

Let $t \in \mathbf{Z}$. By applying the argument used in [8, (1.0.2)] to prove a result of Drozd's, one can easily see that there exist only finitely many positive x in $\mathbf{Z}^n$ with $q_A(x) = t$. We are now going to show that the number of modules in each τ-orbit of Γ_A is finite. Let U and V be modules in the same τ-orbit of Γ_A. Since $\operatorname{End}_A(U)$ and $\operatorname{End}_A(V)$ are division rings by Lemma 2, we conclude from [2, §3] that $\operatorname{End}_A(U) \cong \operatorname{End}_A(V)$. In particular, $q_A([U]) = q_A([V])$. Under our assumption the indecomposable modules in Γ_A are determined by their composition factors [3]. So we see now that each τ-orbit of Γ_A contains only finitely many modules. By Lemma 4 the artin algebra A is of finite representation type.

Combining the above theorem with [7, (4.4)], we get the following immediate consequence.

Corollary. *Let A be an artin algebra such that $\operatorname{mod} A$ contains no short cycle. If there exists a sincere indecomposable module in $\operatorname{mod} A$, then A is a tilted algebra of finite representation type. In particular, A is of global dimension at most two, and the indecomposable A-modules are determined by their composition factors.*

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