

QUANTITATIVE TRANSCENDENCE RESULTS FOR NUMBERS ASSOCIATED WITH LIOUVILLE NUMBERS

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ABSTRACT. In 1937, Franklin and Schneider generalized the Gelfond-Schneider result on the transcendence of α^β . They proved the following theorem: If β is an algebraic, irrational number and α is “suitably well-approximated by algebraic numbers of bounded degree”, then α^β is transcendental. In 1964, Feldman established the algebraic independence of α and α^β under similar conditions. We use results concerning linear forms in logarithms to give quantitative versions of the Franklin-Schneider and Feldman results.

INTRODUCTION

In 1934, Gelfond and Schneider independently solved Hilbert’s seventh problem. They proved that if α is algebraic with $\alpha \log \alpha \neq 0$ and β is an algebraic irrational number, then α^β is transcendental. In 1949, Gelfond generalized the Gelfond-Schneider Theorem to give an algebraic independence result. Transcendence and algebraic independence results have also been given in cases where α is not necessarily algebraic; here we consider the case where α is “well-approximated by algebraic numbers of bounded degree”.

WELL-APPROXIMATED NUMBERS

Given an algebraic number a , we let $H(a)$ denote the maximum absolute value of the coefficients of the minimal polynomial of a over the integers. We let $\deg a$ denote the degree of that polynomial. Given this standard notation, we can quantify the expression “well-approximated by algebraic numbers of bounded degree”.

Suppose that $d_0 \in \mathbb{N}$ and $\Delta : \mathbb{R}^+ \rightarrow \mathbb{R}$ with $\limsup_{T \rightarrow \infty} \Delta(T)/T = \infty$. We say that α is (d_0, Δ) -approximable if for infinitely many natural numbers T there exist algebraic numbers a_T satisfying

$$(0) \quad \deg a_T \leq d_0, \quad H(a_T) \leq \exp(T), \quad 0 < |\alpha - a_T| \leq \exp(-\Delta(T)).$$

In what follows, we let $\{a_{T_j}\}_{j=1}^\infty$ denote a fixed sequence of algebraic numbers satisfying these three conditions.

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Variations of this notation have been used by Brownawell and Waldschmidt [3], Laurent [7], and Tubbs [9]. The conditions given here guarantee that α is transcendental. In fact, it is fairly straightforward to verify that any such α belongs to the class of U -numbers (or U^* -numbers) in the Mahler (respectively, Koksma) classification of transcendental numbers. For more details about U -numbers see [1].

Example. We may obtain transcendental numbers which are “well-approximated by algebraic numbers of arbitrary degree $d_0 \geq 1$ ” by considering ξ^{1/d_0} where

$$\xi = \frac{1}{3} + \sum_{n=1}^{\infty} 10^{-n!}.$$

STATEMENT OF RESULTS

Franklin and Schneider gave the first transcendence result for “well approximated” numbers. Their theorem corresponds to the Gelfond-Schneider result; namely, if α is well approximated by algebraic numbers of bounded degree and β is algebraic but irrational, then α^β is transcendental. In [4] Cijssouw and Waldschmidt gave nice results concerning the simultaneous approximation (by algebraic numbers) of values associated with the exponential function. Their Theorem 2, which follows from strengthened versions of Baker’s lower bounds for linear forms in logarithms of algebraic numbers, can be used to produce a transcendence measure for α^β when α is suitably well approximated and β is an irrational algebraic number. Lower bounds for linear forms are the basis of our results as well. Our first two theorems are quantitative transcendence results for α^β and e^α under suitable approximation hypotheses on α .

In stating our results, we let α be a “well-approximated” complex number with $\alpha \neq 0$, $\alpha \neq 1$, and we let $\{a_{T_j}\}_{j=1}^{\infty}$ denote a fixed sequence of approximations satisfying inequalities (0) for some natural number d_0 and some appropriate function Δ . We let $\log \alpha$ be a fixed (nonzero) determination of the logarithm of α , and for any $z \in \mathbb{C}$ we define $\alpha^z := \exp(z \log \alpha)$. We also let β be an algebraic number of degree d at least two. We consider nonconstant polynomials $P(X) \in \mathbb{Z}[X]$ and $P(X, Y) \in \mathbb{Z}[X, Y]$ of (total) degree D_P and usual height H_P . The letters S_1, \dots, S_4 will denote positive constants which depend only on α , d_0 , Δ , β , and the fixed sequence $\{a_{T_j}\}$ of approximations to α .

Theorem 1. *Let α , $\log \alpha$, d_0 , β , and $P(X)$ be as above, and suppose M satisfies*

$$M \geq D_P^5 (d d_0)^4 (\log H_P + \log D_P) \min\{D_P, H_P\}.$$

Let $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be continuous and strictly increasing with $\lim_{T \rightarrow \infty} f(T)/T \neq \infty$, and suppose that α is $(d_0, T(\log T)f(T))$ -approximable. Then there exists a positive constant S_1 (depending only on α , f , d_0 , β , $\{a_{T_j}\}$ as mentioned above) such that if $\min\{D_P, H_P\} \geq S_1$ then

$$\log |P(\alpha^\beta)| > -\tau M f^{-1}(M) \log(f^{-1}(M))$$

where

$$\tau = \frac{T_k \log T_k}{T_{k-1} \log T_{k-1}}, \quad k = \min\{j : f(T_j) \geq M\}.$$

Theorem 2. Let α , $\log \alpha$, d_0 , and $P(X)$ be as above, and suppose M satisfies

$$M \geq D_P^4 d_0^3 (\log H_P + \log D_P) \min\{D_P, H_P\}.$$

Let $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be continuous and strictly increasing with $\lim_{T \rightarrow \infty} f(T)/T \neq \infty$, and suppose that α is $(d_0, T f(T))$ -approximable. Then there exists a positive constant S_2 such that if $\min\{D_P, H_P\} \geq S_2$ then

$$\log |P(e^\alpha)| > -\tau M f^{-1}(M)$$

where

$$\tau = \frac{T_k}{T_{k-1}}, \quad k = \min\{j : f(T_j) \geq M\}.$$

Using lower bounds for linear forms in logarithms, Feldman [5] gave the first algebraic independence result for (d_0, Δ) -approximable numbers. He showed that there exists a positive constant C_1 such that, if α is $(d_0, C_1 T^2 (\log \log T)^{-1})$ -approximable, then α and α^β are algebraically independent over the rational numbers. Our next result gives a measure of this algebraic independence. Tubbs [9] has given analogous results for the Weierstrass elliptic function.

Theorem 3. Let α , $\log \alpha$, d_0 , β , and $P(X, Y)$ be as above, and suppose N satisfies

$$N \geq D_P^5 d_0^{10} d^4 (D_P + \log H_P) \min\{D_P, H_P\}.$$

Let $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be continuous and strictly increasing with $\lim_{T \rightarrow \infty} g(T)/T \neq \infty$, and suppose that α is $(d_0, T^2 (\log T) g(T))$ -approximable. Then there exists a positive constant S_3 such that if $\min\{D_P, H_P\} \geq S_3$ then

$$\log |P(\alpha, \alpha^\beta)| > -\tau' N (g^{-1}(N))^2 \log(g^{-1}(N))$$

where

$$\tau' = \frac{T_k^2 (\log T_k)}{T_{k-1}^2 (\log T_{k-1})}, \quad k = \min\{j : g(T_j) \geq N\}.$$

We give a similar theorem for α and e^α ; another result of this kind can be found in [10].

Theorem 4. Let α , $\log \alpha$, d_0 , and $P(X, Y)$ be as above, and suppose N satisfies

$$N \geq D_P^4 d_0^8 (D_P + \log H_P) \min\{D_P, H_P\}.$$

Let $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be continuous and strictly increasing with $\lim_{T \rightarrow \infty} g(T)/T \neq \infty$, and suppose that α is $(d_0, T^2 g(T))$ -approximable. Then there exists a positive constant S_4 such that if $\min\{D_P, H_P\} \geq S_4$ then

$$\log |P(\alpha, e^\alpha)| > -\tau' N (g^{-1}(N))^2$$

where

$$\tau' = \frac{T_k^2}{T_{k-1}^2}, \quad k = \min\{j : g(T_j) \geq N\}.$$

AUXILIARY RESULTS

As mentioned previously, the main component of our proofs is a lower bound for certain linear forms in logarithms of algebraic numbers. The particular bound which we use here is due to Philippon and Waldschmidt [8].

To keep the statement of the theorem concise, we introduce some notation first. Given nonzero algebraic numbers $\alpha_1, \dots, \alpha_n$ and $\beta_0, \beta_1, \dots, \beta_n$, we consider the linear form

$$\Lambda = \beta_0 + \beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n.$$

We let D be a positive integer and A_1, \dots, A_n, A, B positive real numbers which satisfy

$$\begin{aligned} D &\geq [\mathbb{Q}(\alpha_1, \dots, \alpha_n, \beta_0, \beta_1, \dots, \beta_n) : \mathbb{Q}], \\ A_j &\geq \max\{H(\alpha_j), \exp(|\log \alpha_j|), e^n\}, \quad 1 \leq j \leq n, \\ A &= \max\{A_1, \dots, A_n, e^e\}, \end{aligned}$$

and

$$B = \max\{H(\beta_j) : 0 \leq j \leq n\}.$$

Proposition 5. *If $\Lambda \neq 0$ then*

$$|\Lambda| \geq \exp(-2^{8n+53} n^{2n} D^{n+2} \log A_1 \dots \log A_n (\log B + \log \log A)).$$

Remark. Recently, Waldschmidt [12] has developed techniques which give stronger lower bounds for these linear forms. These strengthened results may lead to better quantitative results in our setting. While Waldschmidt's new techniques do apply to linear forms with algebraic coefficients (like those we will encounter here), his published results focus on the special case where $\Lambda = b_1 \log \alpha_1 + \dots + b_n \log \alpha_n$ and b_1, \dots, b_n are rational integers. For this reason, we have chosen to use the older (weaker) results here.

The next three lemmas are used in the proofs of both theorems.

Lemma 6. *Suppose $Q(X) \in \mathbb{Z}[X]$ is a nonconstant polynomial of degree at most D_Q and height at most H_Q , and let Θ be any complex number. Then there exists an algebraic number ζ of degree, say, D , and a positive integer m such that*

$$mD \leq D_Q, \quad H(\zeta) \leq (H_Q D_Q)^{1/mD},$$

and

$$|\Theta - \zeta|^m \leq 4^{D_Q^2} (2D_Q H_Q)^{D_Q} |Q(\Theta)|.$$

Proof. See [13, Lemma 2.3].

Lemma 7. *Let v, w be two complex numbers satisfying $|w - e^v| \leq \frac{1}{3}|e^v|$. Then there exists a determination of the logarithm of w such that*

$$|\log w - v| \leq \frac{3}{2} \frac{1}{|e^v|} |w - e^v|.$$

Proof. See [11, Lemma 2.2].

Lemma 8 (Gelfond [6]). *Let P_1, \dots, P_m be polynomials in $\mathbb{C}[X_1, \dots, X_n]$. Suppose the product $P_1 \dots P_m$ has degree D_i in the variable X_i for $i = 1, \dots, n$, and let $D = \sum_{i=1}^n D_i$. Then*

$$H(P_1) \dots H(P_m) \leq e^D H(P_1 \dots P_m).$$

Proof. See [6, Lemma 2, p. 135].

For the proof of Theorem 3 we need an additional lemma.

Lemma 9. Suppose $P(X) \in \mathbb{Z}[X]$ is a polynomial with degree at most D_P and height at most H_P . If $\Theta, \tilde{\Theta} \in \mathbb{C}$ satisfy $|\Theta - \tilde{\Theta}| \leq 1$, then

$$|P(\Theta) - P(\tilde{\Theta})| \leq (|\Theta| + 1)^{D_P-1} H_P D_P^2 |\Theta - \tilde{\Theta}|.$$

Proof. The proof follows if we write the polynomial $P(\Theta) - P(\tilde{\Theta})$ as a sum of differences $(\Theta - \tilde{\Theta})^i$ and then factor $\Theta - \tilde{\Theta}$ from each term in the sum. We use the inequality $|\tilde{\Theta}| \leq |\Theta| + 1$ to estimate the remaining factors.

PROOFS OF THEOREMS 1 AND 3

First we establish Theorem 1.

Proof of Theorem 1. Given $\alpha, \log \alpha, d_0, \beta, P(X), M, f(T)$, and $\{a_{T_j}\}_{j=1}^\infty$ as above, we let $\Delta(T) = T(\log T)f(T)$ and define k and τ as in the statement of the theorem, taking care to choose S_1 sufficiently large so that the definition of k will ensure that k is at least two and, hence, T_{k-1} is defined. We suppose that

$$(1) \quad |P(\alpha^\beta)| \leq \exp(-\tau M f^{-1}(M) \log(f^{-1}(M)))$$

and seek a contradiction.

By Lemma 6 there exists a positive integer $m \leq D_P$ and an algebraic number ζ of degree at most D_P and height at most $H_P D_P$ such that

$$(2) \quad |\zeta - \alpha^\beta|^m \leq 4^{D_P^2} (2D_P H_P)^{D_P} \exp(-\tau M f^{-1}(M) \log(f^{-1}(M))).$$

Substituting for τ and then using the inequality $T_{k-1} < f^{-1}(M)$ (which follows from the minimality of k and the invertibility of f) followed by the lower bound for M , we see that

$$|\zeta - \alpha^\beta|^m \leq \exp(-\tau M f^{-1}(M) \log(f^{-1}(M))/2)$$

provided S_1 is sufficiently large to ensure that $T_k \geq 3$. Taking m th roots and noting that $m \leq D_P$, we have

$$(3) \quad |\zeta - \alpha^\beta| \leq \exp(-\tau M f^{-1}(M) \log(f^{-1}(M))/2D_P).$$

For S_1 sufficiently large, we have $|\zeta - \alpha^\beta| \leq |\alpha^\beta|/3$, so by Lemma 7 there exists a determination of the logarithm of ζ such that

$$(4) \quad |\log \zeta - \beta \log \alpha| \leq c_1 |\zeta - \alpha^\beta|$$

where $c_1 = 3/2|\alpha^\beta|$. Choosing $a = a_{T_k}$ from the sequence of approximations $\{a_{T_j}\}$ and noting that $|\alpha - a| \leq \exp(-\Delta(T_k))$, we see by Lemma 7 (again) that there exists a determination of the logarithm of a such that

$$(5) \quad |\log \alpha - \log a| \leq c_2 |\alpha - a|$$

where $c_2 = 3/2|\alpha|$ (provided that S_1 is chosen sufficiently large to ensure that $\exp(-\Delta(T_k)) \leq |\alpha|/3$).

Now we consider the linear form $\Lambda = \log \zeta - \beta \log a$ in logarithms of algebraic numbers. From the triangle inequality, we have

$$|\Lambda| \leq |\log \zeta - \beta \log \alpha| + |\beta| |\log \alpha - \log a|.$$

Using inequalities (4) and (5) and then inequality (3), and noting that $a = a_{T_k}$ is an approximation to α , we have

$$\begin{aligned} |\Lambda| &\leq c_1 |\zeta - \alpha^\beta| + c_2 |\beta| \cdot |\alpha - a| \\ &\leq c_1 \exp(-\tau M f^{-1}(M) \log(f^{-1}(M))/2D_P) + c_3 \exp(-\Delta(T_k)). \end{aligned}$$

Using the definition of τ and the inequality $T_{k-1} < f^{-1}(M)$ to bound the first term and using the definition of Δ and our choice of k to bound the second term, this last inequality may be reduced to

$$(6) \quad |\Lambda| < \exp(-MT_k(\log T_k)/3D_P)$$

provided S_1 is sufficiently large.

Now we have an upper bound for a linear form in logarithms of algebraic numbers with algebraic coefficients. Since $\alpha \neq 0$, $\alpha \neq 1$, we know that for S_1 (and hence T_k) sufficiently large we have $a_{T_k} \neq 0$ and $a_{T_k} \neq 1$; then a^β is transcendental by the Gelfond-Schneider Theorem. But ζ is algebraic, so the linear form Λ is nonvanishing. This allows us to obtain a lower bound as well, from Proposition 5.

We note that $|\zeta| \leq H(\zeta) + 1 \leq 2H(\zeta)$ and, by virtue of the choice of the logarithm in the proof of Lemma 7, we have $|\log \zeta| \leq c_4$ where c_4 depends only on α , β , and $\log \alpha$. Similarly, $|a| \leq 2H(a)$ and $|\log a| \leq c_5$. Thus we may choose the parameters in Proposition 5 as follows:

$$A_1 = c_6 H_P D_P, \quad A = A_2 = c_6 e^{T_k},$$

$$D = [\mathbb{Q}(\zeta, \beta, a) : \mathbb{Q}] \leq D_P d d_0, \quad B = H(\beta),$$

where $c_6 = \max\{2e^{c_4}, 2e^{c_5}, e^e\}$. Then we have (for S_1 sufficiently large)

$$|\Lambda| \geq \exp(-c_7 2^{73} (D_P d d_0)^4 (\log H_P + \log D_P) T_k \log T_k).$$

Combining this lower bound with our upper bound in (6) yields

$$M < 3c_7 2^{73} D_P^5 (d d_0)^4 (\log H_P + \log D_P).$$

Our lower bound for M leads us to a contradiction and the theorem is established.

The proof of Theorem 2 is similar; the main difference being that the linear form under consideration is $\Lambda = \log \zeta - a$. We establish Theorem 3 next; we omit the proof of Theorem 4.

Proof of Theorem 3. Given α , $\log \alpha$, d_0 , β , $P(X, Y)$, N , $g(T)$, and $\{a_{T_j}\}_{j=1}^\infty$ as above, we let $\Delta(T) = T^2(\log T)g(T)$ and define k and τ' as in the statement of the theorem, taking care to choose S_3 sufficiently large as in the previous proof. We let

$$G(N) = \tau' N (g^{-1}(N))^2 (\log g^{-1}(N))$$

and suppose that

$$(7) \quad \log |P(\alpha, \alpha^\beta)| \leq -G(N).$$

If $\deg_X P = 0$, then we obtain an immediate contradiction by taking $Q(Y) = P(X, Y)$. Supposing the result is false, we have

$$\log |Q(\alpha^\beta)| \leq -G(N).$$

By applying Theorem 1 with $M = N$ and $f(T) = g(T)$ we have

$$\log |Q(\alpha^\beta)| > -\tau N g^{-1}(N) \log(g^{-1}(N))$$

which, together with the upper bound for $|Q(\alpha^\beta)|$, leads to $T_k < T_{k-1}$, a contradiction as desired.

If $\deg_X P > 0$, we choose $a = a_{T_k}$ from a fixed sequence of approximations to α and let $a_1 = a, a_2, \dots, a_{\delta_0}$ be the conjugates of a over \mathbb{Q} . Then $\delta_0 \leq d_0$. We also let $q \in \mathbb{Z}$ be a denominator of a and note that $|q| \leq e^{T_k}$. Then we consider the new polynomial

$$Q(Y) = q^{\delta_0 D_P} \prod_{i=1}^{\delta_0} P(a_i, Y).$$

Since the product on the right-hand side is fixed under σ in the Galois group of $\mathbb{Q}(a)$ over \mathbb{Q} , we know that $\prod_{i=1}^{\delta_0} P(a_i, Y) \in \mathbb{Q}[Y]$. Because we multiply by an appropriate denominator, we then have $Q(Y) \in \mathbb{Z}[Y]$.

Furthermore,

$$(8) \quad \deg Q \leq D_P \delta_0 \leq D_P d_0$$

and the height H_Q of the polynomial Q satisfies

$$(9) \quad \begin{aligned} H_Q &\leq |q|^{\delta_0 D_P} H_P^{\delta_0} \max\{1, |a_1|, \dots, |a_{\delta_0}|\}^{\delta_0 D_P} (1 + 2D_P)^{\delta_0} \\ &\leq e^{T_k d_0 D_P} H_P^{d_0} \max\{1, |a_1|, \dots, |a_{\delta_0}|\}^{d_0 D_P} (1 + 2D_P)^{d_0}. \end{aligned}$$

We also know that

$$\max\{1, |a_1|, \dots, |a_{\delta_0}|\} \leq 1 + H(a) = 1 + H(a_{T_k}) \leq e^{2T_k},$$

so, for S_3 sufficiently large, inequality (9) may be reduced to

$$(10) \quad H_Q \leq \exp(4d_0(D_P + \log H_P)T_k).$$

Eventually, we will estimate $|Q(\alpha^\beta)|$. First we want to verify that Q is a nonzero polynomial. To do this, we will establish that $P(a_i, Y)$ is a nonzero polynomial for each $i = 1, \dots, \delta_0$.

If $P(a_i, Y)$ is identically zero for some $i \in \{1, \dots, \delta_0\}$, then we see that the minimal polynomial for a over \mathbb{Z} must divide $P(X, Y)$ and Lemma 8 shows that $H(a) \leq e^{D_P} H_P$. Without loss of generality, we may assume that each approximation a_{T_j} satisfies $H(a_{T_j}) > e^{T_j-1}$. Defining k in terms of this (possibly new) sequence, we have $H(a) = H(a_{T_k}) > e^{T_k-1}$; therefore,

$$(11) \quad T_k - 1 < D_P + \log H_P.$$

On the other hand, our choice of T_k shows that for S_3 sufficiently large we have $T_k > 2(D_P + \log H_P)$, contradicting the inequality in (11). Hence, for every i ($1 \leq i \leq \delta_0$), the polynomial $P(a_i, Y)$ is nonzero, and therefore $Q(Y)$ is nonzero as well.

To bound $|Q(\alpha^\beta)|$, we estimate $|P(a_i, \alpha^\beta)|$ for $i = 1, \dots, \delta_0$. For $i = 1$, we have (from the triangle inequality)

$$|P(a, \alpha^\beta)| \leq |P(a, \alpha^\beta) - P(\alpha, \alpha^\beta)| + |P(\alpha, \alpha^\beta)|.$$

Using Lemma 9 and our assumption (7), we get

$$\begin{aligned} |P(a, \alpha^\beta)| &\leq H_P(1 + D_P)^3 \max\{1, |\alpha^\beta|\}^{D_P} (1 + |\alpha|)^{D_P} |a - \alpha| + \exp(-G(N)) \\ &\leq \exp(-\Delta(T_k)/2) + \exp(-G(N)). \end{aligned}$$

Using our definitions of Δ , k , G , and τ' (as in the proof of Theorem 1), we may reduce this last inequality to

$$|P(a, \alpha^\beta)| \leq \exp(-NT_k^2(\log T_k)/3).$$

For $i = 2, \dots, \delta_0$, we bound term by term to get

$$(12) \quad |P(a_i, \alpha^\beta)| \leq H_P(1 + D_P)^2 \max\{1, |a_i|\}^{D_P} \max\{1, |\alpha^\beta|\}^{D_P}.$$

But $|a_i| \leq H(a) + 1 \leq e^{2T_k}$, so (12) becomes

$$\begin{aligned} |P(a_i, \alpha^\beta)| &\leq H_P(1 + D_P)^2 (e^{2T_k})^{D_P} \max\{1, |\alpha^\beta|\}^{D_P} \\ &\leq \exp(3(D_P + \log H_P)T_k) \end{aligned}$$

for S_3 sufficiently large.

Taking the product over i and multiplying by the denominator $q^{\delta_0 D_P}$, we have

$$\begin{aligned} |Q(\alpha^\beta)| &\leq |q|^{\delta_0 D_P} \prod_{i=1}^{\delta_0} |P(a_i, \alpha^\beta)| \\ &< \exp(T_k d_0 D_P + 3(\delta_0 - 1)(D_P + \log H_P)T_k - NT_k^2(\log T_k)/3) \\ &\leq \exp(-NT_k^2(\log T_k)/4). \end{aligned}$$

For S_3 sufficiently large, the new polynomial $Q(Y)$ is nonconstant. For if it were constant, then $Q(\alpha^\beta) \in \mathbb{Z} - \{0\}$. But for S_3 sufficiently large, we also have $|Q(\alpha^\beta)| < 1$, which would lead to a contradiction.

This allows us to use Lemma 6 to find a positive integer $m \leq d_0 D_P$ and an algebraic number ζ of degree at most $d_0 D_P$ and height at most $H_Q d_0 D_P$ such that

$$(13) \quad |\zeta - \alpha^\beta|^m \leq 4^{(d_0 D_P)^2} (2d_0 D_P H_Q)^{d_0 D_P} |Q(\alpha^\beta)|.$$

Using inequality (10) to estimate H_Q , we see that

$$H(\zeta) \leq H_Q d_0 D_P \leq \exp(5d_0(D_P + \log H_P)T_k)$$

and inequality (13) reduces to

$$|\zeta - \alpha^\beta|^m < \exp(-NT_k^2(\log T_k)/5).$$

Taking m th roots, we have

$$(14) \quad |\zeta - \alpha^\beta| < \exp(-NT_k^2(\log T_k)/5D_P d_0).$$

As in the previous proof, Lemma 7 (applied twice) yields determinations of $\log \zeta$ and $\log a$ which satisfy

$$(15) \quad |\log \zeta - \beta \log \alpha| \leq c_9 |\zeta - \alpha^\beta|, \quad |\log \alpha - \log a| \leq c_{10} |\alpha - a|$$

where c_9 and c_{10} depend only on $\beta \log \alpha$. Using the triangle inequality, along with inequalities (14) and (15), the approximation property of α , and our choice of k , we have (for S_3 sufficiently large)

$$(16) \quad |\log \zeta - \beta \log a| < \exp(-NT_k^2(\log T_k)/7D_P d_0).$$

As in the proof of Theorem 1, for S_3 sufficiently large, this linear form $\Lambda = \log \zeta - \beta \log a$ is nonvanishing and we may apply Proposition 5 to obtain a lower bound. As before, $\exp(|\log \zeta|) \leq c_{11}$ and $\exp(|\log a|) \leq c_{12}$ and we may choose the parameters as follows:

$$A = A_1 = c_{13} \exp(5d_0(D_P + \log H_P)T_k), \quad A_2 = c_{13} \exp(T_k),$$

$$D = [\mathbb{Q}(\zeta, a, \beta) : \mathbb{Q}] \leq D_Q d_0 d \leq D_P d_0^2 d, \quad B = H(\beta).$$

Then we have

$$|\Lambda| \geq \exp(-c_{14} 2^{73} D_P^4 d_0^9 d^4 (D_P + \log H_P) T_k^2 \log T_k)$$

for S_3 sufficiently large by our choice of T_k .

Combining this lower bound with the upper bound of line (16) gives the desired contradiction and the result is established.

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