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A NOTE ON A TYPE OF APPROXIMATE IDENTITY IN THE FOURIER ALGEBRA

BRIAN FORREST AND MAHATHEVA SKANTHARAJAH

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ABSTRACT. Let $A(G)$ denote the Fourier algebra of the locally compact group G. We show that for a large class of groups an ideal I in $A(G)$ has a Δ-weak bounded approximate identity if and only if it has a bounded approximate identity.

Let $\mathscr{A}$ denote a commutative Banach algebra with structure space $\Delta(\mathscr{A})$. A net $\{u_\alpha\}_{\alpha\in\mathscr{U}}$ in $\mathscr{A}$ is called a Δ-weak bounded approximate identity if there exists $M<\infty$ such that $\|u_\alpha\|_{\mathscr{A}}\le M$ for every $\alpha\in\mathscr{U}$ and if

$$\lim_\alpha|\varphi(u_\alpha u)-\varphi(u)|=0 \quad \text{for every } u\in\mathscr{A}\,,\ \varphi\in\Delta(\mathscr{A})\,.$$

Δ-weak approximate identities were introduced in [5], where they were called weak bounded approximate identities, a term that is often used for another class of bounded approximate identities. The authors showed that it is possible for an algebra $\mathscr{A}$ to have a Δ-weak bounded approximate identity but not a bounded approximate identity. In this note, we show that this is not the case for ideals in the Fourier algebra of a large class of locally compact groups.

Let G be a locally compact group. Let $A(G)$ denote the Fourier algebra of G as defined by Eymard in [1]. $A(G)$ is a commutative Banach algebra of continuous functions on G with structure space G. Given a closed subset A of G, we define the ideal $I(A)$ as follows:

$$I(A)=\{u\in A(G)|u(x)=0 \text{ for every } x\in A\}\,.$$

Given an ideal I of $A(G)$, we define

$$Z(I)=\{x\in G|u(x)=0 \text{ for every } u\in I\}\,.$$

We say that $A\in\mathscr{R}_c(G)$ if A is closed in G and $A\in\mathscr{R}(G_d)$, the ring of subsets of G generated by all left cosets of subgroups of G_d, where G_d denotes G with the discrete topology.

Theorem 1. *Let I be a closed ideal in $A(G)$. If I has a Δ-weak bounded approximate identity, then $Z(I)\in\mathscr{R}_c(G)$.*

Proof. Let $\{u_\alpha\}_{\alpha\in\mathscr{U}}$ be a Δ-weak bounded approximate identity in I. Let $A=Z(I)$. If $X\in G\backslash A$, then there exists $u\in I$ with $\operatorname{supp} u\cap A=\varnothing$ and

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$u(x) = 1$. Since $\lim_\alpha u_\alpha u(x) = u(x) = 1$, it follows that $\{u_\alpha\}_{\alpha\in\mathscr{U}}$ converges pointwise to $1_{G\setminus A}$. However, $u_\alpha \in B(G_d)$, the Fourier-Stieltjes algebra of G_d, and $\{u_\alpha\}_{\alpha\in\mathscr{U}}$ is bounded in $B(G_d)$. It follows from [1, Corollary 2, p. 202] that $1_{G\setminus A} \in B(G_d)$ and, hence, that $1_A \in B(G_d)$. The result now follows from Host's noncommutative version of Cohen's idempotent theorem [4]. □

The next result follows immediately from Theorem 1 and [2, Theorem 3.11].

Corollary 2. *Let G be an amenable $[SIN]$-group. Let I be a closed ideal in $A(G)$. Then the following are equivalent:*

(i) *I has a Δ-weak bounded approximate identity.*
(ii) *I has a bounded approximate identity.*
(iii) $Z(I) \in \mathscr{R}_C(G)$.

We answer a question raised in [4, p. 157].

Corollary 3. *Let G be a locally compact abelian group. Let I be a closed ideal in $L^1(G)$. Then I has a Δ-weak bounded approximate identity if and only if $Z(I) \in \mathscr{R}_C(G)$.*

Finally, we remark that for discrete groups the existence of a Δ-weak bounded approximate identity in $A(G)$ can easily be shown to imply the amenability of G. Consequently, by verifying that the proof of [2, Theorem 3.20] holds for Δ-weak bounded approximate identities, we see that for discrete G an ideal I of $A(G)$ has a Δ-weak bounded approximate identity if and only if it has a bounded approximate identity. Such ideals can be completely characterized in terms of $Z(I)$ (see [2, Theorem 3.20]).

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Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1