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A_∞-CONDITION FOR THE JACOBIAN OF A QUASI-CONFORMAL MAPPING

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ABSTRACT. We show that the Jacobian J_f of a quasi-conformal mapping $f\colon \mathbf{B}^n \to D$ is an A_∞-weight in $\mathbf{B}^n$ if and only if D is a John domain. A similar question concerning $J_{f^{-1}}$ is also studied.

1. INTRODUCTION

Suppose that $f\colon D \to D'$ is a K-quasi-conformal mapping between domains D and D' in $\mathbf{R}^n$, $n \geq 2$. Then Gehring's [G] well-known result ensures that the Jacobian J_f of f satisfies the reverse Hölder inequality

$$(1.1) \qquad \left(\fint_Q J_f^p \, dx \right)^{1/p} \leq C \fint_Q J_f \, dx$$

for some $p = p(n, K) > 1$ and $C = C(n, K) \geq 1$ whenever Q is a cube in D such that $2Q \subset D$. In other words, J_f satisfies Muckenhoupt's A_∞ condition when restricted to "Whitney cubes" in D. In general, one cannot hope that (1.1) holds true for all cubes $Q \subset D$, and in this note we address the question: when is J_f an A_∞ weight in D? That is, when does (1.1) hold for all cubes $Q \subset D$? We shall write $J_f \in A_\infty(D)$ if the answer is affirmative; the A_∞ data (p, C) in (1.1) is denoted by $\|J_f\|_{A_\infty(D)}$.

For the most part, we consider the case when either D or D' is $\mathbf{B}^n$, the unit ball of $\mathbf{R}^n$. Then our main result is the following.

1.2. **Theorem.** *Let $f\colon \mathbf{B}^n \to D$ be a K-quasi-conformal mapping. Then $J_f \in A_\infty(\mathbf{B}^n)$ if and only if D is a c-John domain with center $f(0)$. Moreover, $\|J_f\|_{A_\infty(\mathbf{B}^n)}$ and c depend only on each other and on n and K.*

Recall that a domain D in $\mathbf{R}^n$ is a *c-John domain with center $x_0 \in D$* if there is $c \geq 1$ such that each point $x \in D$ can be joined to x_0 by an arc γ

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satisfying

$$\operatorname{diam} \gamma[x, y] \le c \operatorname{dist}(y, \partial D)$$

for all $y \in \gamma$, where $\gamma[x, y]$ designates the subarc of γ between x and y.

It was recently established in [AK] that, if $f: \mathbf{B}^n \to D$ is a quasi-conformal mapping, then $J_f \in L^{1+\varepsilon}(\mathbf{B}^n)$ for some $\varepsilon > 0$ if and only if D satisfies a *quasi-hyperbolic boundary condition.* Theorem 1.2 is another example in this vein, where the geometry of the target reflects the analytic properties of the mapping.

Since every A_∞-weight in a ball is the restriction of an A_∞-weight in $\mathbf{R}^n$ (see [Ho]), we obtain

1.3. **Corollary.** *Let $f: \mathbf{B}^n \to D$ be a quasi-conformal mapping. There is an A_∞-weight w in $\mathbf{R}^n$ such that $J_f = w|\mathbf{B}^n$ if and only if D is a John domain.*

Another corollary is obtained by combining Theorem 1.2 with the fact that every BMO function in $\mathbf{B}^n$ with a small BMO-norm is the logarithm of an A_∞-weight (see [GCRF, p. 409]).

1.4. **Corollary.** *Let f be a quasi-conformal mapping of $\mathbf{B}^n$ into $\mathbf{R}^n$. There exists a constant $\nu(n) > 0$ such that*

$$\|\log J_f\|_{\mathrm{BMO}(\mathbf{B}^n)} < \nu(n)$$

implies that $f(\mathbf{B}^n)$ is a John domain.

In fact, we can choose $\nu(n) = \ln 2/2^{n+2}$, the constant appearing in the John-Nirenberg lemma.

It follows from a theorem of Reimann [R] that $\log J_f$ belongs to $\mathrm{BMO}(\mathbf{B}^n)$ for any quasi-conformal mapping f of $\mathbf{B}^n$ (see also [S]). By Corollary 1.4 a small BMO norm restricts the image. Astala and Gehring [AG2, 1.5B] proved that, when $n = 2$, in the presence of a small BMO norm the image will be a quasi-disk, provided $1 \le K < 2$. Take notice that in Corollary 1.4 no restriction on K is imposed.

In general, for a quasi-conformal mapping $f: D \to D'$, the condition $J_f \in A_\infty(D)$ places a more severe constraint on the target domain D' than it does on D (cf. [AK]). This can be seen in §3, where we give a simple geometric condition on D ensuring $J_f \in A_\infty(D)$ for all quasi-conformal mappings $f: D \to \mathbf{B}^n$. This condition is much weaker than being a John domain, and it shows that $J_f \in A_\infty(D)$ does not imply $J_{f^{-1}} \in A_\infty(D')$ in general. We also give an example which shows that the condition is essentially sharp.

We remark that John domains which can be mapped quasi-conformally onto a ball now admit more than ten essentially different characterizations. See [V4, NV, He] and for $n = 2$ [GHM, P, Z].

After this paper was submitted we found out that an equivalent formulation of Theorem 1.2 for $n = 2$ had been proved in [Z, p. 158]. We thank Michel Zinsmeister for bringing this reference to our attention. We also thank the referee for a careful reading of our paper.

Preliminaries. Throughout, Q will denote an open n-cube with $\mathrm{l}(Q)$ its side length. We also write Q_x for a cube centered at x. If $\lambda > 0$, then λQ denotes a cube with the same center and $\mathrm{l}(\lambda Q) = \lambda \mathrm{l}(Q)$. The closure of Q is $\overline{Q}$. The Lebesgue n-measure of a set E is $|E|$ and $\fint_E g\,dx = (1/|E|)\int_E g\,dx$ stands for the integral average of a function g in E.

We shall be somewhat cavalier in dealing with the various constants appearing in the proofs, but always careful in pointing out the dependence in the statements of the theorems. The expression $a \approx b$ means that there is a constant C such that $C^{-1}a \le b \le Ca$.

By a *Whitney cube* in a domain D we mean a cube $Q \subset D$ such that $\operatorname{dist}(Q, \partial D)/4 \le l(Q) \le 4\operatorname{dist}(Q, \partial D)$ (note that this terminology is not standard). Then if $f\colon D \to D'$ is a K-quasi-conformal mapping and Q is a Whitney cube in D, we have

$$(1.5)\qquad \left(\int_Q J_f\,dx\right)^{1/n} \approx \operatorname{diam} fQ \approx \operatorname{dist}(fQ, \partial D'),$$

where the constants depend only on n and K; see [V1, 18.1, 33.3]. Moreover, there is a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$, depending only on n and K, such that the local quasi-symmetry condition

$$(1.6)\qquad |a-x| \le t|b-x| \Rightarrow |f(a)-f(x)| \le \eta(t)|f(b)-f(x)|$$

holds whenever a, b, x lie in a Whitney cube Q; see [V2, 2.4]. Both (1.5) and (1.6) will be used repeatedly in this paper.

For the basic properties of A_∞-weights we refer to [CF, GCRF].

2. Proof of Theorem 1.2

In the following string of lemmata, $f\colon D \to D'$ is a K-quasi-conformal mapping between proper subdomains D and D' of $\mathbf{R}^n$.

2.1. **Lemma.** *$J_f \in A_\infty(D)$ if and only if there is $C \ge 1$ such that*

$$\int_Q J_f\,dx \le C\int_{Q/2} J_f\,dx$$

for all cubes $Q \subset D$. Moreover, $\|J_f\|_{A_\infty(D)}$ and C depend only on each other and on n and K.

Proof. The necessity is immediate because A_∞-weights are doubling. To treat the sufficiency, we apply Hölder's inequality and Gehring's "local" result (1.1) to obtain

$$\begin{aligned}\fint_Q J_f\,dx \le C\fint_{Q/2} J_f\,dx &\le C\left(\fint_{Q/2} J_f^{1/n}\,dx\right)^{\delta n}\left(\fint_{Q/2} J_f^p\,dx\right)^{(1-\delta)/p}\\ &\le C\left(\fint_Q J_f^{1/n}\,dx\right)^{\delta n}\left(\fint_Q J_f\,dx\right)^{1-\delta},\end{aligned}$$

where $1 = \delta n + (1-\delta)/p$. Thus for all cubes $Q \subset D$ we have the reverse Hölder inequality

$$\fint_Q J_f\,dx \le C\left(\fint_Q J_f^{1/n}\,dx\right)^n,$$

which is well known to yield (1.1) [G]. The lemma follows.

2.2. Lemma. *$J_f \in A_\infty(D)$ if and only if there is $C \geq 1$ such that*

$$\max_{y\in\partial Q_x} |f(x) - f(y)| \leq C \min_{y\in\partial Q_x/2} |f(x) - f(y)|$$

for all cubes Q_x such that $\overline{Q}_x \subset D$. Moreover, $\|J_f\|_{A_\infty(D)}$ and C depend only on each other and on n and K.

Proof. The sufficiency follows from Lemma 2.1. Indeed,

$$\begin{aligned}\int_{Q_x} J_f\,dx = |fQ_x| &\leq C\left(\max_{y\in\partial Q_x} |f(x) - f(y)|\right)^n \\ &\leq C|f\tfrac{1}{2}Q_x| = C\int_{Q_x/2} J_f\,dx\end{aligned}$$

whenever Q_x is compactly contained in D, and the general case follows from the continuity of the integral.

To prove the necessity, fix $Q = Q_x$ such that $\overline{Q} \subset D$ and let $z \in \partial Q$ be a point such that

$$|f(x) - f(z)| = \max_{y\in\partial Q} |f(x) - f(y)|.$$

Next, let $Q_1, Q_2, \dots$ be a sequence of disjoint Whitney cubes in Q such that Q_1 has the center x and the union $\bigcup_j Q_j$ forms a "tower" with z as a limit point. More precisely (assuming that the line segment $[x, z]$ lies, say, on the nth coordinate axis), the sides of Q_j are parallel to the coordinate axes, the centers of Q_j lie on $[x, z]$, and Q_{j+1} is placed on the top of Q_j. Then

$$|Q| \approx |Q_1| \quad\text{and}\quad |Q_j| \approx |Q_{j+1}| \approx \left|\bigcup_{i=j}^{\infty} Q_i\right|.$$

Hence, the assumption $J_f \in A_\infty(D)$ implies

$$\sum_{i=1}^{\infty} \int_{Q_i} J_f\,dx \leq C\int_{Q_1} J_f\,dx$$

(see [CF, Lemma 5]), and similarly for each $j > 1$

$$\sum_{i=j}^{\infty} \int_{Q_i} J_f\,dx \leq C\int_{Q_j} J_f\,dx,$$

where C does not depend on j. Because $\int_{Q_i} J_f\,dx \approx (\operatorname{diam} fQ_i)^n$, we find that there is a constant C such that $\sum_{i=j}^{\infty}(\operatorname{diam} fQ_i)^n \leq C(\operatorname{diam} fQ_j)^n$ for each $j \geq 1$. An elementary lemma on infinite series (see [AG1, 3.1]) now implies

$$\sum_{i=j}^{\infty} \operatorname{diam} fQ_i \leq C \operatorname{diam} fQ_j,$$

whence,

$$|f(x) - f(z)| \leq \sum_{i=1}^{\infty} \operatorname{diam} fQ_i \leq C \operatorname{diam} fQ_1.$$

Finally since $\operatorname{diam} fQ_1 \approx |f(x) - f(y)|$ for all $y \in \partial\frac{1}{2}Q$ by (1.6), the proof is complete.

2.3. **Lemma.** *$J_f \in A_\infty(D)$ if and only if fQ_x is a c-John domain with center $f(x)$ for each cube $Q_x \subset D$. Moreover, $\|J_f\|_{A_\infty(D)}$ and c depend only on each other and on n and K.*

Proof. The necessity is proved as in Lemma 2.2. Namely, it is clearly sufficient to verify that

$$\sup_{y \in \partial Q_z} |f(z) - f(y)| \le c \operatorname{dist}(f(z), \partial D)$$

whenever $\overline{Q}_z \subset D$. Then the image of the line segment $[x, y]$ under f can be chosen for the John arc γ joining $f(y)$ to the John center $f(x)$ in fQ_x. As for the sufficiency, let $Q = Q_x \subset D$. Because fQ is c-John with center $f(x)$, we have $\operatorname{diam} fQ \le 2c \operatorname{dist}(f(x), \partial fQ)$. On the other hand, $\operatorname{dist}(f(x), \partial fQ) \approx \operatorname{diam} f\frac{1}{2}Q$ and hence

$$\int_Q J_f\, dx = |fQ| \le C(\operatorname{diam} fQ)^n \le C(\operatorname{diam} f\tfrac{1}{2}Q)^n \le C \int_{Q/2} J_f\, dx,$$

where (1.5) was used in the last inequality. Thus the assertion follows from Lemma 2.1.

The necessity part in Theorem 1.2 easily follows from Lemma 2.3. To prove the sufficiency, we invoke a recent result of Väisälä [V4, 2.20] which implies that every K-quasi-conformal mapping f of $\mathbf{B}^n$ onto a c-John domain D with center $f(0)$ satisfies the following quasi-symmetry condition:

$$|a - x| \le t|b - x| \Rightarrow \delta_D(f(a), f(x)) \le \eta(t)\delta_D(f(b), f(x))$$

whenever $a, b, x \in \mathbf{B}^n$. Here $\delta_D(z, w)$ is the *internal distance* of z and w in D defined as the infimum of the diameters of all arcs joining z and w in D, and η (as in (1.6)) depends only on n, K, and c.

Thus if $f\colon \mathbf{B}^n \to D$ is K-quasi-conformal and D is c-John with center $f(0)$, we have that

$$|f(x) - f(z)| \le \delta_D(f(x), f(z)) \le C \min_{y \in \partial Q_x/2} |f(x) - f(y)|$$

whenever $Q_x \subset \mathbf{B}^n$ and $z \in \partial Q$. Now an application of Lemma 2.2 establishes the sufficiency in Theorem 1.2.

3. Mappings onto the unit ball

In this section we seek conditions on D that guarantee $J_f \in A_\infty(D)$ for each quasi-conformal mapping $f\colon D \to \mathbf{B}^n$. A fairly simple sufficient geometric criterion can be given, which we next describe.

3.1. **Condition** $P(\delta, x_0)$. Suppose that D is a bounded domain in $\mathbf{R}^n$. Let $x_0 \in D$ be a *center point* of D, i.e., $\operatorname{dist}(x_0, \partial D) \ge \operatorname{dist}(x, \partial D)$ for all $x \in D$. We say that D has *property* $P(\delta, x_0)$ if there is $\delta \in (0, (2\sqrt{n})^{-1})$ such that every cube $Q \subset D$ satisfies: either $x_0 \in (1+\delta)Q$ or else there is a cube $Q' \subset D$, centered at a point $w \in \partial Q$, such that $\mathrm{l}(Q') = \delta\, \mathrm{l}(Q)$ and that $Q' \setminus (1 + \delta/2)\overline{Q}$ lies in the x_0-component of $D \setminus (1 + \delta/2)\overline{Q}$.

3.2. **Theorem.** *Suppose that f is a K-quasi-conformal mapping of a bounded domain D onto $\mathbf{B}^n$ such that $f(x_0) = 0$, where x_0 is a center point of D. If D has property $P(\delta, x_0)$, then $J_f \in A_\infty(D)$. Moreover, $\|J_f\|_{A_\infty(D)}$ depends only on n, K, and δ.*

For the proof, we require

3.3. **Subinvariance.** If $f: D \to \mathbf{B}^n$ is a K-quasi-conformal mapping and if $G \subset D$ is a c-uniform domain, then $f(G) \subset \mathbf{B}^n$ is a c'-uniform domain with c' depending only on n, c, and K. This follows from [FHM, pp. 120–121] combined with [V3, 5.6].

Recall that a domain D is *c-uniform* if there is $c \geq 1$ such that each pair of points $x, y \in D$ can be joined by an arc γ satisfying $\operatorname{diam} \gamma \leq c|x - y|$ and

$$\min\{\operatorname{diam} \gamma[x, z], \operatorname{diam} \gamma[y, z]\} \leq c \operatorname{dist}(z, \partial D)$$

for all $z \in \gamma$, where $\gamma[w, z]$ designates the subarc of γ between w and z.

We denote by $\operatorname{mod}(F_1, F_2; A)$ the usual conformal modulus of the family of all curves joining two disjoint continua F_1 and F_2 in A. Then

$$\varphi_n(t) \leq \operatorname{mod}(F_1, F_2; \mathbf{R}^n) \leq \mu_n(t), \tag{3.4}$$

where

$$t = \frac{\operatorname{dist}(F_1, F_2)}{\min\{\operatorname{diam} F_1, \operatorname{diam} F_2\}}$$

and $\varphi_n, \mu_n: (0, \infty) \to (0, \infty)$ are decreasing homeomorphisms. See, for instance, [V1, 11.9; GM, 2.6; Vu, II.7].

Proof of Theorem 3.2. By Lemma 2.2 it suffices to show that there is C such that

$$\max_{y \in \partial Q_x} |f(x) - f(y)| \leq C \min_{y \in \partial Q_x/2} |f(x) - f(y)| \tag{3.5}$$

whenever $\overline{Q}_x \subset D$. Fix such a cube $Q = Q_x$; by the local quasi-symmetry of f (see (1.6) or [V2, 2.4]), we may assume that $(1 + \delta)Q$ is not contained in D. Suppose first that $x_0 \in (1 + \delta)Q$. Then $\operatorname{dist}(x_0, Q) \leq l(Q)/4$ while $\operatorname{dist}(x_0, \partial D) \geq \operatorname{dist}(x, \partial D) \geq l(Q)/2$, and it is easily verified that $G = Q \cup B$ is a $c(n)$-uniform domain in D, where B is the largest ball centered at x_0 and contained in D. In particular, fG is a $c(n, K)$-uniform domain in $\mathbf{B}^n$ by the subinvariance 3.3, and applying (1.5) and (1.6) we deduce that $\operatorname{diam} fG \leq C \operatorname{dist}(f(x_0), \partial fG)$. Now a quasi-conformal mapping between bounded uniform domains satisfies (1.6) for any triple a, b, x with η depending only on the constants of uniformity, n, K, and the location of the image of a center point; see [V3, 5.6] or [V4, 2.20]. Thus (3.5) follows if $x_0 \in (1 + \delta)Q$.

Next assume that $x_0 \in D \setminus (1 + \delta)Q$. Let $z \in \partial Q$ be a point with

$$|f(x) - f(z)| = \max_{y \in \partial Q} |f(x) - f(y)|,$$

and let L be the line segment from x to z. Owing to condition $P(\delta, x_0)$ and modulus estimate (3.4), one can construct an arc J joining a point $w \in \partial \frac{1}{2} Q$ to x_0 in D in such a manner that

$$\operatorname{mod}(L, J; D) \leq c = c(\delta, n) < \infty.$$

The K-quasi-conformality of f implies

$$\begin{aligned}\varphi_n\left(\frac{\operatorname{dist}\{fL, fJ\}}{\min\{\operatorname{diam} fL, \operatorname{diam} fJ\}}\right) &\le \operatorname{mod}(fL, fJ; \mathbf{R}^n)\\ &\le 2\operatorname{mod}(fL, fJ; \mathbf{B}^n) \le 2Kc\end{aligned}$$

(for the middle inequality, see, e.g., [GM]), and hence

$$\min\{\operatorname{diam} fL, \operatorname{diam} fJ\} \le C\operatorname{dist}\{fL, fJ\} \le C|f(x) - f(w)|.$$

On the other hand, since $(1+\delta)Q \not\subset D$ and since $\operatorname{diam} J \ge \mathrm{l}(Q)/2$, one easily infers that $\operatorname{diam} fJ \ge c > 0$, where c depends only on n and K. Thus

$$|f(x) - f(z)| \le \operatorname{diam} fL \le C|f(x) - f(w)|,$$

and because $|f(x) - f(w)| \approx |f(x) - f(y)|$ for all $y \in \partial\frac{1}{2}Q$ by (1.6), the proof is complete.

We have not found a simple geometric criterion that would characterize the domains D for which $J_f \in A_\infty(D)$ for each quasi-conformal mapping $f\colon D \to \mathbf{B}^n$. However, the following example illustrates the sharpness of condition $P(\delta, x_0)$.

3.6. **Example.** Suppose $n = 2$. Let $x_0 = 0$ and $x_i = x_{i-1} + 2^{-i}$, $i = 1, 2, \dots$, be points on the positive real axis, and let Q_i be the open cube centered at x_i with $\mathrm{l}(Q_i) = 2^{-i-2}$ and sides parallel to the coordinate axes. Connect the cubes Q_{i-1} and Q_i by a rectangle $R_i = \{(x, y)\colon x_{i-1} < x < x_i, -\varepsilon_i < y < \varepsilon_i\}$, where $\varepsilon_i < 2^{-i-3}$. Set $D = \bigcup_i(Q_i \cup R_i)$. Then D is a bounded simply connected domain that satisfies condition $P(\delta, 0)$ for some $\delta > 0$ if and only if $\liminf \varepsilon_i 2^i > 0$ as $i \to \infty$. We show that, if $\liminf \varepsilon_i 2^i = 0$, then $J_f \notin A_\infty(D)$ for a conformal mapping $f\colon D \to \mathbf{B}^2$ with $f(0) = 0$.

To this end, fix $\varepsilon > 0$. We denote by $c(\varepsilon)$ any function that depends on ε and $c(\varepsilon) \to 0$ as $\varepsilon \to 0$. Choose $i > 1$ such that $\varepsilon_i 2^i < \varepsilon$, and write $Q = Q_i$, $R = R_i$, and $x = x_i$. Let $I = \partial Q \cap R$. Then the harmonic measure $\omega = \omega(fI, fQ)$ satisfies $\omega(f(x)) = \omega(I, Q)(x) < c(\varepsilon)$. By subinvariance, $fQ \subset \mathbf{B}^2$ is a simply connected uniform domain and, hence, a quasi-disk [MS]. Now fI is a cross-cut in $\mathbf{B}^2$ separating $f(x)$ from the origin, and standard harmonic measure estimates in quasidisks (see, e.g., [P, Theorem 1]) imply

$$\rho \equiv \operatorname{dist}(f(x), \partial fQ) \le \operatorname{dist}(f(x), fJ) \le c(\varepsilon)\operatorname{dist}(f(x), fI),$$

where J is the arc on ∂Q, complementary to I. Therefore,

$$\rho^2 \le c(\varepsilon)|fQ| = c(\varepsilon)\int_Q J_f\,dx.$$

On the other hand, if Q' is a cube, centered at $f(x)$ with $\mathrm{l}(Q') = \rho/2$, then $f^{-1}Q'$ contains a cube λQ for some absolute constant $0 < \lambda < 1$. Thus if the Jacobian J_f were an A_∞-weight in D, we would have

$$\lambda^2 = \frac{|\lambda Q|}{|Q|} \le C\left(\frac{\int_{\lambda Q} J_f\,dx}{\int_Q J_f\,dx}\right)^\alpha \le C\left(\frac{\rho^2}{|fQ|}\right)^\alpha \le c(\varepsilon),$$

which is a contradiction as $\varepsilon \to 0$.

4. Concluding remarks

In light of Lemma 2.3, one might ask if $\mathbf{B}^n$ could be replaced by a more general domain in Theorems 1.2 and 3.2. It follows from [He, 3.1, 7.1; NV, 3.6, 3.9] that fQ is a John domain for all cubes $Q \subset D$ whenever $f: D \to D'$ is a quasi-conformal mapping of any domain D onto a *broad domain* D'. (Broad domains were introduced by Väisälä in [V4], and the definition involves a condition for the moduli of certain curve families; for instance, a John domain is broad if it is quasi-conformally equivalent to a uniform domain.) However, this does not automatically guarantee that $J_f \in A_\infty(D)$ for all quasi-conformal mappings $f: D \to D'$ onto a broad domain D' because the constants generally depend on the ratio

$$\frac{\operatorname{diam} fQ}{\operatorname{dist}(f(x), \partial D)},$$

where x is the center of Q. The condition $P(\delta, x_0)$ in Theorem 3.2 is needed to control this quantity. Consequently, many relations between uniform domains, John domains, broad domains, and the A_∞ condition remain unknown.

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