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LOWER BOUNDS FOR RELATIVE CLASS NUMBERS OF CM-FIELDS

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ABSTRACT. Let $\mathbf{K}$ be a CM-field that is a quadratic extension of a totally real number field $\mathbf{k}$. Under a technical assumption, we show that the relative class number of $\mathbf{K}$ is large compared with the absolute value of the discriminant of $\mathbf{K}$, provided that the Dedekind zeta function of $\mathbf{k}$ has a real zero s such that $0 < s < 1$. This result will enable us to get sharp upper bounds on conductors of totally imaginary abelian number fields with class number one or with prescribed ideal class groups.

Let $\mathbf{K}$ be a CM-field that is a quadratic extension of a totally real number field $\mathbf{k}$.

If the Dedekind zeta function of $\mathbf{K}$ is nonpositive at some s_0 that belongs to the interval $]0, 1[$, then it is well known that we can get good lower bounds for the residue at $s = 1$ of this zeta function and for the relative class number of $\mathbf{K}$. In Proposition A, we give explicit forms of such a result. They will enable us to consider in Corollary c the class number one problem for cyclotomic fields in a more efficient way than those one can find in the literature (see [7, 12]).

Now, if the Dedekind zeta function of $\mathbf{k}$ has a zero in $]0, 1[$, then in Theorem 1 we give lower bounds for the relative class number of $\mathbf{K}$. Our proof assumes the technical assumption $d(\mathbf{K}) \geq 4^N d(\mathbf{k})^2$ where N is the degree of $\mathbf{k}$ and where $d(\mathbf{k})$ and $d(\mathbf{K})$ are the absolute values of the discriminants of $\mathbf{k}$ and $\mathbf{K}$.

Let us stress that, under this previous technical assumption, one remarkable consequence of Theorem 1 is that the zeta function of $\mathbf{k}$ has no real zeros in the open interval $]0, 1[$ provided that the relative class number of $\mathbf{K}$ is less than or equal to 2 (or provided that this relative class number is not "too large"). If we can deduce from this that the zeta function of $\mathbf{K}$ is nonpositive on this interval, then from our previous lower bounds for the relative class numbers we may get very good upper bounds on the discriminants of $\mathbf{K}$, provided that the relative class number of $\mathbf{K}$ is less than or equal to 2 (or provided that this relative class number if not "too large").

Our main application of these techniques is the proof in Corollary b that the zeta function of the real quadratic subfield $\mathbf{k}$ of a totally imaginary cyclic

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quartic number field $\mathbf{K}$ with ideal class group of exponent ≤ 2 and discriminant $d(\mathbf{K})$ has no real zero in the interval $1 - 2/\operatorname{Log}(d(\mathbf{K})) \leq s < 1$. This is the result we needed in [5] in order to prove that there are exactly 33 such quartic number fields.

Let $\mathbf{k}$ be an algebraic number field with class number $h(\mathbf{k})$ and regulator $\operatorname{Reg}(\mathbf{k})$. Let $d(\mathbf{k})$ be the absolute value of the discriminant of $\mathbf{k}$. Set

$$[\mathbf{k}\colon \mathbf{Q}] = r_1 + 2r_2, \qquad A = 2^{-r_2} d(\mathbf{k})^{1/2}\pi^{-(r_1+2r_2)/2},$$

$$\lambda(\mathbf{k}) = \frac{2^{r_1} h(\mathbf{k}) \operatorname{Re} g(\mathbf{k})}{w(\mathbf{k})} \quad \text{where } w(\mathbf{k}) \text{ is the number of roots of unity of } \mathbf{k},$$

$$F_{\mathbf{k}}(s) = A^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_{\mathbf{k}}(s),$$

so that $F_{\mathbf{k}}(s)$ has a simple pole at $s = 1$ with residue equal to $\lambda(\mathbf{k})$.

Whenever $y \in (\mathbf{R}_+^*)^{r_1+r_2}$, we set

$$\|y\|_{\mathbf{k}} = (y_1 \cdot \cdots \cdot y_{r_1})(y_{r_1+1} \cdot \cdots \cdot y_{r_1+r_2})^2, \text{ so that } \|\lambda y\|_{\mathbf{k}} = \lambda^N \|y\|, \lambda > 0;$$
$$Tr_{\mathbf{k}}(y) = y_1 + \cdots + y_{r_1} + 2(y_{r_1+1} + \cdots + y_{r_1+r_2}), \text{ so that } Tr_{\mathbf{k}}(\lambda y) = \lambda y.$$

It is well known (see [4]) that we have the integral representation

$$F_{\mathbf{k}}(s) = \frac{\lambda(\mathbf{k})}{s(s-1)} + I_{\mathbf{k}}(s)$$

with

(1)
$$I_{\mathbf{k}}(s) = \sum_{\mathbf{B}} \int_{\|y\|_{\mathbf{k}} \geq 1} \exp(-\pi d(\mathbf{k})^{-1/N} N_{\mathbf{k}/\mathbf{Q}}(\mathbf{B})^{2/N} Tr_{\mathbf{k}}(y))[\|y\|_{\mathbf{k}}^{s/2} + \|y\|_{\mathbf{k}}^{(1-s)/2}]\frac{dy}{y},$$

where the sum is taken over all integral ideals $\mathbf{B} \neq 0$ of $\mathbf{k}$.

From now on, we assume that s is a real number such that $\frac{1}{2} \leq s < 1$.

For $y = (y_1, \ldots, y_N) \in (\mathbf{R}_+^*)^N$ we set $Tr(y) = y_1 + \cdots + y_N$ and $\|y\| = y_1 \cdot \cdots \cdot y_N$.

If $\mathbf{K}$ is a totally imaginary number field of degree $2N$ that is a quadratic extension of a totally real number field $\mathbf{k}$ of degree N, then $I_{\mathbf{K}}(s)$ and $I_{\mathbf{k}}(s)$ are integrals in $(\mathbf{R}_+^*)^N$ and we have:

$$Tr_{\mathbf{K}}(y) = 2Tr(y) \text{ and } \|y\|_{\mathbf{K}} = \|y\|^2, \text{ so that } \|y\|_{\mathbf{K}} \geq 1 \text{ if and only if } \|y\| \geq 1;$$
$$Tr_{\mathbf{k}}(y) = Tr(y) \text{ and } \|y\|_{\mathbf{k}} = \|y\|, \text{ so that } \|y\|_{\mathbf{k}} \geq 1 \text{ if and only if } \|y\| \geq 1.$$

Moreover, we have the natural injection map $i_{\mathbf{K}/\mathbf{k}}$ from the group of fractional ideals of $\mathbf{k}$ in the group of fractional ideals of $\mathbf{K}$ that satisfies

$$N_{\mathbf{K}/\mathbf{Q}}(i_{\mathbf{K}/\mathbf{k}}(\mathbf{B}))^{2/2N} = N_{\mathbf{k}/\mathbf{Q}}(\mathbf{B})^{2/N}$$

whenever $\mathbf{B}$ is an integral ideal of $\mathbf{k}$. We thus get

$$(2)\quad I_{\mathbf{K}}(s) \geq \sum_{\mathbf{B}} \int_{\|y\| \geq 1} \exp(-2\pi d(\mathbf{K})^{-1/2N} N_{\mathbf{k}/\mathbf{Q}}(\mathbf{B})^{2/N} Tr(y))[\|y\|^s + \|y\|^{1-s}]\frac{dy}{y}$$

and

(3)
$$I_{\mathbf{k}}(s) = \sum_{\mathbf{B}} \int_{\|y\| \geq 1} \exp(-\pi d(\mathbf{k})^{-1/N} N_{\mathbf{k}/\mathbf{Q}}(\mathbf{B})^{2/N} Tr_{\mathbf{k}}(y))[\|y\|^{s/2} + \|y\|^{(1-s)/2}]\frac{dy}{y},$$

where the sums are taken over all integral ideals $\mathbf{B} \neq 0$ of $\mathbf{k}$.

Hence, if we assume that we have $d(\mathbf{K}) \geq 4^N d(\mathbf{k})^2$, noticing that we have $\|y\|^2 > \|y\|$ whenever $\|y\| > 1$, then (2) and (3) provide us with

$$\text{(a)} \qquad I_{\mathbf{K}}(s) > I_{\mathbf{k}}(s).$$

Moreover, from (2) we have

$$(4) \qquad I_{\mathbf{K}}(s) \geq \sum_{\mathbf{B}} \int_{\|y\| \geq 1} \exp(-2\pi d(\mathbf{K})^{-1/2N} N_{\mathbf{k}/\mathbf{Q}}(\mathbf{B})^{2/N} Tr(y)) \|y\|^s \frac{dy}{y}$$

and from (3) we have

$$(5) \qquad I_{\mathbf{k}}(s) \leq 2 \sum_{\mathbf{B}} \int_{\|y\| \geq 1} \exp(-\pi d(\mathbf{k})^{-1/N} N_{\mathbf{k}/\mathbf{Q}}(\mathbf{B})^{2/N} Tr(y)) \|y\|^{s/2} \frac{dy}{y},$$

where the sums are taken over all integral ideals $\mathbf{B} \neq 0$ of $\mathbf{k}$.

We change variables in (4), making the multiplicative translation $y = d(\mathbf{K})^{1/2N} Y / 2d(\mathbf{k})^{1/N}$. We note that, under the hypothesis $d(\mathbf{K}) \geq 4^N d(\mathbf{k})^2$, the domain $\|Y\| \geq 1$ is included in the domain $\|Y\| \geq 2^N d(\mathbf{k})/\sqrt{d(\mathbf{K})}$. Using $\|Y\|^s > \|Y\|^{s/2}$ whenever $\|Y\| > 1$, we get

$$\text{(b)} \qquad I_{\mathbf{K}}\left(\frac{1}{2}\right) > \left(\frac{d(\mathbf{K})}{4^N d(\mathbf{k})^2}\right)^{1/4} I_{\mathbf{k}}\left(\frac{1}{2}\right),$$

$$\text{(c)} \qquad I_{\mathbf{K}}(s) > \frac{1}{2}\left(\frac{d(\mathbf{K})}{4^N d(\mathbf{k})^2}\right)^{s/2} I_{\mathbf{k}}(s), \qquad \frac{1}{2} < s < 1.$$

Now, as $\mathbf{K}$ is a CM-field of degree $2N$ that is a quadratic extension of a totally real number field $\mathbf{k}$ of degree N, it is well known that we have

$$\frac{\lambda(\mathbf{K})}{\lambda(\mathbf{k})} = \frac{h^*(\mathbf{K})}{Qw(\mathbf{K})},$$

where $h^*(\mathbf{K})$ is the relative class number of $\mathbf{K}$ and where $Q = 1$ or 2 (see [12, Theorem 4.12]). Moreover, if $\frac{1}{2} \leq s_0 < 1$ is a real zero of $\zeta_{\mathbf{k}}$, then we have $\zeta_{\mathbf{K}}(s_0) = 0$ since $\mathbf{K}/\mathbf{k}$ is normal, so that $F_{\mathbf{k}}(s_0) = F_{\mathbf{K}}(s_0) = 0$, so that

$$\frac{\lambda(\mathbf{K})}{\lambda(\mathbf{k})} = \frac{I_{\mathbf{K}}(s_0)}{I_{\mathbf{k}}(s_0)}.$$

Hence, we get the following theorem whose assertion (b) is much more precise than the one given in [6] (note that as soon as the totally real number field $\mathbf{k}$ is fixed, then there are only finitely many totally imaginary number fields $\mathbf{K}$ that are quadratic extensions of $\mathbf{k}$ and such that $d(\mathbf{K}) < 4^N d(\mathbf{k})^2$):

Theorem 1. *Let* $\mathbf{K}$ *be a CM-field of degree* $2N$ *that is a quadratic extension of a totally real number field* $\mathbf{k}$ *of degree* N. *Let us suppose that we have* $d(\mathbf{K}) \geq 4^N d(\mathbf{k})^2$.

If the Dedekind zeta function of $\mathbf{k}$ *has a real zero* s_0 *such that* $\frac{1}{2} \leq s_0 < 1$, *then we have the three following lower bounds for the relative class number* $h^*(\mathbf{K})$

of $\mathbf{K}$:

$$\text{(a)}\quad h^*(\mathbf{K}) > Qw(\mathbf{K}) \geq 2,$$

$$\text{(b)}\quad h^*(\mathbf{K}) > \frac{Qw(\mathbf{K})}{\sqrt{2^N d(\mathbf{k})}} d(\mathbf{K})^{1/4} \quad \textit{if } s_0 = \frac{1}{2},$$

$$\text{(c)}\quad h^*(\mathbf{K}) > \frac{1}{2} Qw(\mathbf{K}) \left(\frac{d(\mathbf{K})}{4^N d(\mathbf{k})^2}\right)^{s_0/2} \quad \textit{if } \frac{1}{2} < s_0 < 1.$$

Hence, the zeta function of $\mathbf{k}$ *has no real zero in the interval* $1 - 2/\mathrm{Log}(d(\mathbf{K})) \leq s < 1$ *provided that we have* $h^*(\mathbf{K}) \leq \sqrt{d(\mathbf{K})}/e2^N d(\mathbf{k})$.

Remark. Theorem 1 does not apply to the class number one problem for cyclotomic fields, for $d(\mathbf{K}) = d(\mathbf{k})^2$ whenever $\mathbf{K} = \mathbf{Q}(\zeta_n)$ with n not a prime power, and $d(\mathbf{K}) = pd(\mathbf{k})^2$ whenever $\mathbf{K} = \mathbf{Q}(\zeta_{p^a})$, with p an odd prime. Nevertheless, in Corollary c, we will manage to consider the class number one problem for cyclotomic fields (with prime powers conductors). Theorem 1 does not apply to the class number one problem for totally imaginary biquadratic abelian number fields with group $(\mathbf{Z}/2\mathbf{Z})^2$, for $d(\mathbf{K}) = d(\mathbf{k})^2$ whenever $\mathbf{K} = \mathbf{Q}(\sqrt{-p}, \sqrt{-q})$, p and q prime and congruent to 3 mod 4.

In fact, Theorem 1 applies nicely to class numbers problems for totally imaginary cyclic number fields with bounded degrees.

Theorem 2. *Let* $\mathbf{K}$ *be a CM-field of degree* $2N$ *that is a quadratic extension of a totally real number field* $\mathbf{k}$ *of degree* N. *Let* $\mathrm{Res}_1(\zeta_{\mathbf{k}})$ *be the residue at* $s = 1$ *of the Dedekind zeta function* $\zeta_{\mathbf{k}}$ *of* $\mathbf{k}$. *Let us suppose that the Dedekind zeta function* $s \mapsto \zeta_{\mathbf{K}}(s)$ *satisfies*

$$\zeta_{\mathbf{K}}(1 - 2/\mathrm{Log}(d(\mathbf{K}))) \leq 0.$$

Then, we have the following lower bounds for the relative class number of $\mathbf{K}$:

$$h^*(\mathbf{K}) \geq f(N, \mathbf{K}) \frac{1}{\mathrm{Res}_1(\zeta_{\mathbf{k}})} \frac{2Qw(\mathbf{K})}{e(2\pi)^N} \frac{\sqrt{d(\mathbf{K})/d(\mathbf{k})}}{\mathrm{Log}(d(\mathbf{K}))},$$

with the two possible choices:

$$\text{(a)}\qquad f(N, \mathbf{K}) = 1 - \frac{2\pi N e^{1/N}}{d(\mathbf{K})^{1/2N}}$$

or

$$\text{(b)}\qquad f(N, \mathbf{K}) = \frac{2}{5} \exp\left(-\frac{2\pi N}{d(\mathbf{K})^{1/2N}}\right), \quad \textit{whenever } N \geq 2.$$

Proof. We get the desired result from Proposition A thanks to

$$\frac{\mathrm{Res}_1(\zeta_{\mathbf{K}})}{\mathrm{Res}_1(\zeta_{\mathbf{k}})} = (2\pi)^N \sqrt{\frac{d(\mathbf{k})}{d(\mathbf{K})} \frac{\lambda(\mathbf{K})}{\lambda(\mathbf{k})}} = (2\pi)^N \frac{h^*(\mathbf{K})}{Qw(\mathbf{K})} \sqrt{\frac{d(\mathbf{k})}{d(\mathbf{K})}}.$$

Proposition A. *Let* $\mathbf{K}$ *be a totally imaginary number field of degree* $2N$. *If its Dedekind zeta function* $s \mapsto \zeta_{\mathbf{K}}(s)$ *is such that* $\zeta_{\mathbf{K}}(s_0) \leq 0$ *for some* s_0 *real in* $[\frac{1}{2}, 1[$, *then we have the following effective lower bounds for the residue at* $s = 1$ *of this zeta function*:

$$\text{(a)}\qquad \mathrm{Res}_1(\zeta_{\mathbf{K}}) \geq (1 - s_0) d(\mathbf{K})^{(s_0-1)/2} \left\{1 - \frac{2\pi N}{d(\mathbf{K})^{s_0/2N}}\right\};$$

$$\text{(b)}\quad \mathrm{Res}_1(\zeta_{\mathbf{K}}) \geq \frac{2}{5}(1 - s_0) d(\mathbf{K})^{(s_0-1)/2} \exp\left(-\frac{2\pi N}{d(\mathbf{K})^{1/2N}}\right), \quad \textit{whenever } N \geq 2.$$

Proof. From (1) where we use only the term of the sum corresponding to the ideal $\mathbf{B}$ equal to the ring of algebraic integers of $\mathbf{K}$, and where we disregard the term with $\|y\|_{\mathbf{K}}^{(1-s)/2}$, we get

$$\frac{\sqrt{d(\mathbf{K})}}{(2\pi)^N}\frac{\mathrm{Res}_1(\zeta_{\mathbf{K}})}{s_0(1-s_0)} = \frac{\lambda(\mathbf{K})}{s_0(1-s_0)} \geq \int_{\|y\|\geq 1} \exp(-2\pi d(\mathbf{K})^{-1/2N}Tr(y))\|y\|^{s_0}\frac{dy}{y}.$$

Setting $y = d(\mathbf{K})^{1/2N}Y$, we get

$$\begin{aligned}(6)\quad \frac{\mathrm{Res}_1(\zeta_{\mathbf{K}})}{(2\pi)^N} &\geq s_0(1-s_0)d(\mathbf{K})^{(s-1)/2}\int_{\|Y\|\geq d(\mathbf{K})^{-1/2}} \exp(-2\pi Tr(Y))\|Y\|^{s_0}\frac{dY}{Y} \\ &= (1-s_0)d(\mathbf{K})^{(s_0-1)/2}\{f(s_0) - J_{\mathbf{K}}(s_0)\}\end{aligned}$$

with

$$f(s) = s\left[\frac{\Gamma(s)}{(2\pi)^s}\right]^N \quad \text{and} \quad J_{\mathbf{K}}(s) = s\int_{\|Y\|\leq d(\mathbf{K})^{-1/2}} \exp(-2\pi Tr(Y))\|Y\|^{s}\frac{dY}{Y}.$$

Since $\{Y;\|Y\| \leq d(\mathbf{K})^{-1/2}\}$ is included in $\{Y;\exists i \in \{1,\dots,N\}/Y_i \leq d(\mathbf{K})^{-1/2N}\}$, we have (using $e^{-2\pi y} \leq 1$, $y \geq 0$)

$$J_{\mathbf{K}}(s) \leq Ns\left[\frac{\Gamma(s)}{(2\pi)^s}\right]^{N-1}\int_0^{d(\mathbf{K})^{-1/2N}} e^{-2\pi y}y^s\frac{dy}{y} \leq Nf(s)\frac{(2\pi)^s}{s\Gamma(s)}d(\mathbf{k})^{-s/2N}.$$

Hence,

$$\frac{\mathrm{Res}_1(\zeta_{\mathbf{K}})}{(2\pi)^N} \geq (1-s_0)d(\mathbf{K})^{(s_0-1)/2}f(s_0)\left\{1 - N\frac{(2\pi)^{s_0}}{s_0\Gamma(s_0)}d(\mathbf{K})^{-s_0/2N}\right\}.$$

Since $s \mapsto f(s)$ decreases on $]0,1[$, we have $f(s_0) \geq f(1) = (1/2\pi)^N$. Since $s \mapsto (2\pi)^s/s\Gamma(s)$ increases on $]0,1[$, we get the desired first result. In order to get the second desired result, we start from (6) and use the third point of the following lemma with $x = d(\mathbf{K})^{-1/2N}$ (so that from the Minkowski's lower bound $d(\mathbf{K})^{1/2N} \geq \pi N^2/((2N)!)^{1/N}$ we have $x \leq \frac{1}{2}$, $2N \geq 4$):

Lemma. *Set $P_N(t) = \sum_{n=0}^{N-1} t^n/n!$. Then,*

(i)
$$\int_{Tr(Y)\geq t} \exp(-Tr(Y))\,dY = P_N(t)e^{-t}, \qquad N \geq 1.$$

(ii)
$$\int_{Tr(Y)\leq t} \exp(-Tr(Y))\,dY = 1 - P_N(t)e^{-t}, \qquad N \geq 1.$$

(iii)
$$\int_{1\geq\|y\|\geq x^N} \exp(-2\pi Tr(y))\,dy = \frac{e^{-2\pi Nx}}{(2\pi)^N}(1 - P_N(2\pi N(1-x))e^{-2\pi N(1-x)}).$$

(iv) $x_N = 1 - P_N(\pi N)e^{-\pi N}$ *is an increasing sequence that converges towards* 1, *so that* $x_N \geq \frac{4}{5}$, $N \geq 2$.

Proof. Part (iii) is proved from (ii) using the fact that the domain $\{y; y \in (\mathbf{R}_+^*)^N,\ y_i \geq x \text{ and } N \geq Tr(y)\}$ is included in the domain $\{y; y \in (\mathbf{R}_+^*)^N$ and

$1 \geq \|y\| \geq x^N\}$ and changing variables making the translation $y_i = x + Y_i/2\pi$. Part (iv) follows from the inequality $P_{N+1}((N+1)\pi) \leq e^\pi P_N(N\pi)$. Indeed, we have

$$\begin{aligned} P_{N+1}((N+1)\pi) &= \sum_{n=0}^{N} \frac{(\pi N)^n}{n!}\left(\frac{N+1}{N}\right)^n \leq \left(\frac{N+1}{N}\right)^N \sum_{n=0}^{N} \frac{(\pi N)^n}{n!} \\ &\leq e\left(P_N(N\pi) + \frac{(N\pi)^N}{N!}\right) = e\left(P_N(N\pi) + \pi\frac{(N\pi)^{N-1}}{(N-1)!}\right) \\ &\leq e(1+\pi)P_N(N\pi). \quad \square \end{aligned}$$

Remark. Theorems 1 and 2 apply nicely to the determination of CM-fields $\mathbf{K}$ with "small" class numbers, provided that the fields $\mathbf{K}$ are CM-fields that are quadratic extensions of totally real number fields $\mathbf{k}$ such that $\zeta_{\mathbf{K}}/\zeta_{\mathbf{k}}$ is nonnegative on $]0, 1[$. Indeed, Theorem 1 then implies that the zeta functions $\zeta_{\mathbf{k}}$ have no real zero in the interval $1 - 2/\operatorname{Log}(d(\mathbf{K})) \leq s < 1$. Hence, $\zeta_{\mathbf{k}}(s_0) < 0$ and $\zeta_{\mathbf{K}}(s_0) \leq 0$ with $s_0 = 1 - 2/\operatorname{Log}(d(\mathbf{K}))$, so that Theorem 2 provides us with good lower bounds for $h^*(\mathbf{K})$. Since we seek "small" class numbers, this will provide us with upper bounds on $d(\mathbf{K})$.

Let us point out that these assumptions "$\zeta_{\mathbf{K}}/\zeta_{\mathbf{k}}$ are nonnegative on $]0, 1[$" are satisfied as soon as the number fields $\mathbf{K}$ are totally imaginary and cyclic over $\mathbf{Q}$ and such that 4 divides $[\mathbf{K}\colon \mathbf{Q}] = 2N$, for $\zeta_{\mathbf{K}}/\zeta_{\mathbf{k}}$ is then a product of L-functions that come in conjugate pairs.

For example, we first give the following corollary, which greatly improves upon the upper bounds given in [1] or [10]:

Corollary a. *Let $\mathbf{K}$ be a cyclic quartic totally imaginary number field with conductor f and class number $h(\mathbf{K})$. If $h(\mathbf{K}) = 1$ then $f \leq 4500$. If $h(\mathbf{K}) = 2$ then $f \leq 10000$.*

Let $\mathbf{K}$ be a cyclic octic totally imaginary number field with conductor f and class number $h(\mathbf{K})$. If $h(\mathbf{K}) = 1$ then $f = 32$ or f is prime and $f \leq 3000$.

Proof. We only prove the first point. Let $\mathbf{k}$ be the real quadratic subfield of $\mathbf{K}$, let $f_{\mathbf{k}}$ be the conductor of $\mathbf{k}$, and let $L(s, \chi_{f_{\mathbf{k}}})$ be the L-function of $\mathbf{k}$. First, $f_{\mathbf{k}}$ divides f, so that we have $f_{\mathbf{k}} \leq f$. Moreover, $d(\mathbf{k}) = f_{\mathbf{k}}$ and $d(\mathbf{K}) = f_{\mathbf{k}} f^2$. Hence, $d(\mathbf{K})/4^2 d(\mathbf{k})^2 = f^2/16 f_{\mathbf{k}}$ is greater than or equal to 1 as soon as we have $f \geq 16$. Hence, from Theorem 1(a) we deduce that the Dedekind zeta function of $\mathbf{k}$ has no zero on the interval $[\frac{1}{2}, 1[$ as soon as $h^*(\mathbf{K}) = 1$ or 2, provided that we have $f \geq 16$. Since the Dedekind zeta function of $\mathbf{K}$ can be written $\zeta_{\mathbf{K}}(s) = \zeta_{\mathbf{k}}(s)|L(s, \chi_f)|^2$, $s \in]0, 1[$, we can apply Theorem 2. Since $\operatorname{Res}_1(\zeta_{\mathbf{k}}) = L(1, \chi_{f_{\mathbf{k}}}) \leq \frac{1}{2}\operatorname{Log}(f_{\mathbf{k}}) + 1 \leq \frac{1}{2}\operatorname{Log}(f) + 1$ (see [8, Lemma 8.4]) and since $5f^2 \leq d(\mathbf{K}) \leq f^3$, Theorem 2 provides us with the following lower bound from which we get the desired results:

$$h(\mathbf{K}) \geq h^*(\mathbf{K}) \geq \frac{2}{3e\pi^2}\left(1 - \frac{4\pi e^{1/2}}{(5f^2)^{1/4}}\right)\frac{f}{(\operatorname{Log}(f) + 2)\operatorname{Log}(f)}. \quad \square$$

Corollary b. *Let $\mathbf{K}$ be a cyclic quartic totally imaginary number field with conductor f. Then the Dedekind zeta function of the real quadratic subfield $\mathbf{k}$ of $\mathbf{K}$ has no zero in the interval $[1 - 2/\operatorname{Log}(d(\mathbf{K})), 1[$ provided that the ideal class group of $\mathbf{K}$ is of exponent ≤ 2.*

Proof. If the ideal class group of $\mathbf{K}$ is of exponent ≤ 2, then $\mathbf{k}$ is principal and $f_{\mathbf{k}} = 8$ or $f_{\mathbf{k}}$ is prime and such that $f_{\mathbf{k}} \equiv 1 \pmod 4$. Conversely, if $f_{\mathbf{k}} \equiv 1 \pmod 4$ is prime, if $\mathbf{k}$ is principal, and if we define f_2 by means of $f = f_{\mathbf{k}} f_2$, then the ideal class group of $\mathbf{K}$ has 2-rank $t-1$ where t is the number of prime ideals that ramify in the quadratic extension $\mathbf{K}/\mathbf{k}$. Hence, $t \leq 1+2\omega(f_2)$ where $\omega(f_2)$ is the number of prime divisors of f_2 (the proofs of these assertions can be found in [5]). Let us suppose that the Dedekind zeta function of $\mathbf{k}$ had a real zero s_0 such that $1 - 2/\operatorname{Log}(d(\mathbf{K})) \leq s_0 < 1$. Then, as $d(\mathbf{K}) = f_{\mathbf{k}} f^2 = f_{\mathbf{k}}^3 f_2^2$ and $d(\mathbf{k}) = f_{\mathbf{k}}$, Theorem 1 would imply $4^{\omega(f_2)} \geq h(\mathbf{K}) \geq h^*(\mathbf{K}) \geq \sqrt{f_{\mathbf{k}}} f_2/4e$.

Now, $f_{\mathbf{k}} \geq 211$ implies $\sqrt{f_{\mathbf{k}}}/4e > 4/3$, so that we would have $4^{\omega(f_2)} > 4f_2/3$. Since 4 divides f_2 as soon as f_2 is even, this inequality is never satisfied. On the other hand, if $5 \leq f_{\mathbf{k}} \leq 211$, then $s \mapsto L(s, \chi_{f_{\mathbf{k}}})$ has no zero on $]0, 1[$ (see [9]). Thus, we get the desired result. □

Lower bounds for the relative class numbers of cyclotomic fields. Now, we would like to show that Theorem 2 applies to CM-fields with unbounded degrees. For example, we show that Theorem 2(b) enables us to get good upper bounds on the conductors of the cyclotomic fields (with prime-power conductors) with relative class numbers equal to 1. We first give a less tedious proof and more precise form of Lemma 11.5 of Washington [12]; i.e., we give an upper bound on $\operatorname{Res}_1(\mathbf{k})$ with $\mathbf{k}$ being the maximal totally real subfield of a cyclotomic field with prime power conductor.

We define $g(b) = b - 1 + H(b, 1)$, where

$$H(b, s) = \sum_{n\geq 0} \left(\frac{1}{(n+b)^s} - \frac{1}{(n+1)^s} \right), \qquad \begin{cases} \operatorname{Re}(s) > 0, \\ b > 0. \end{cases}$$

Whenever $\chi: \mathbf{N} \mapsto \mathbf{C}$ is a complex-valued function which is periodic mod m, such that $\chi(m) = 0$ and $\sum_{a=1}^{m-1} \chi(a) = 0$, we have

$$L(s, \chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s} = \frac{1}{m^s} \sum_{a=1}^{m-1} \chi(a) H\left(\frac{a}{m}, s\right).$$

Consequently, whenever χ_m is a nontrivial (not necessarily primitive) even Dirichlet character mod m we have $\sum_{a=1}^{m-1} a\chi_m(a) = 0$ and

$$L(1, \chi_m) = \frac{1}{m} \sum_{a=1}^{m-1} \chi(a) g\left(\frac{a}{m}\right).$$

Lemma (i). *$g(b) \geq 0$ and $g(b)^2 + g(b)g(1-b) \leq 1/b^2$, $0 < b < 1$.*

Proof. Follows from the following two inequalities:

$$g(b) = b - 1 + \frac{1}{b} - 1 + \sum_{n\geq 1} \left(\frac{1}{n+b} - \frac{1}{n+1} \right) \geq b + \frac{1}{b} - 2 = \frac{(1-b)^2}{b} \geq 0,$$

$$g(b) = b - 1 + \frac{1}{b} - \sum_{n\geq 1} \frac{b}{n(n+b)} \leq b - 1 + \frac{1}{b} - \sum_{n\geq 1} \frac{b}{n(n+1)} = \frac{1}{b} - 1. \quad \square$$

Lemma (ii). $\left|\prod_{\chi_m \text{ even}, \chi_m \neq 1} L(1, \chi_m)\right| \leq (\pi^2/6)^{(\phi(m)-2)/4}$, *where the product is over the (not necessarily primitive) even Dirichlet characters* mod m.

Proof.

$$\sum_{\substack{\chi_m \text{ even} \\ \chi_m \neq 1}} \chi_m(a)\overline{\chi}_m(b) = \begin{cases} \phi(m)/2 - 1 & \text{if } a \equiv \pm b (\operatorname{mod} m) \\ & \quad \text{and } \operatorname{GCD}(a, m) = 1, \\ 0 \text{ or } -1 & \text{otherwise.} \end{cases}$$

Hence, by the arithmetic-geometric mean inequality,

$$\begin{aligned} &\left(\prod_{\substack{\chi_m \text{ even} \\ \chi_m \neq 1}} |L(1, \chi_m)|^2 \right)^{2/(\phi(m)-2)} \\ &\quad \leq \frac{2m^{-2}}{\phi(m) - 2} \sum_{a=1}^{m-1} \sum_{b=1}^{m-1} g\left(\frac{a}{m}\right) g\left(\frac{b}{m}\right) \left(\sum_{\substack{\chi_m \text{ even} \\ \chi_m \neq 1}} \chi_m(a)\overline{\chi}_m(b) \right) \\ &\quad \leq \frac{1}{m^2} \sum_{\substack{a=1 \\ (a,m)=1}}^{m-1} g\left(\frac{a}{m}\right)^2 + g\left(\frac{a}{m}\right) g\left(1 - \frac{a}{m}\right) \leq \sum_{\substack{a=1 \\ (a,m)=1}}^{m-1} \frac{1}{a^2} \\ &\quad \leq \sum_{\substack{a=1 \\ (a,m)=1}}^{+\infty} \frac{1}{a^2} = \frac{\pi^2}{6} \prod_{\substack{p \text{ prime} \\ p \text{ divides } m}} \left(1 - \frac{1}{p^2}\right) \leq \frac{\pi^2}{6}. \quad \square \end{aligned}$$

Now, in order to apply Theorem 2 to the cyclotomic case, and thanks to the fact that the Dedekind zeta function of a CM cyclic number field factorises on $]0, 1[$ into a product of L-functions that come in conjugate pairs, apart from the two L-functions associated to the principal character and to some quadratic character, we must find an explicit zero-free region for an L-function associated to a quadratic character.

Lemma (iii). *Let* χ *be a quadratic nonprincipal character* $\operatorname{mod} f$. *Set* $N = \frac{1}{2}\varphi(f)$. *Then, for* $\sigma \geq 1/\operatorname{Log}(3)$ *we have*

$$|L'(\sigma, \chi)| \leq \sum_{n=2}^{N+2} \frac{\operatorname{Log}(n)}{n^\sigma} \leq (N+2)^{1-\sigma} \sum_{n=2}^{N+2} \frac{\operatorname{Log}(n)}{n}.$$

Proof. We have

$$L'(\sigma, \chi) = -\chi(2)\frac{\operatorname{Log}(2)}{2^\sigma} - \sum_{k \geq 0} \left(\sum_{n=kf+3}^{(k+1)f+2} \chi(n) \frac{\operatorname{Log}(n)}{n^\sigma} \right).$$

Now, $n \mapsto \operatorname{Log}(n)/n^\sigma$ decreases for $n \geq 3$ provided that we have $\sigma \geq 1/\operatorname{Log}(3)$. Moreover, in each set of f consecutive integers there are N of them such that $\chi(n) = -1$ and N of them such that $\chi(n) = +1$. Hence, for each $k \geq 0$ we have

$$\left| \sum_{n=kf+3}^{(k+1)f+2} \chi(n) \frac{\operatorname{Log}(n)}{n^\sigma} \right| \leq u_k - v_k,$$

with

$$u_k = \sum_{n=kf+3}^{kf+N+2} \frac{\mathrm{Log}(n)}{n^\sigma} \quad \text{and} \quad v_k = \sum_{n=(k+1)f+3-N}^{(k+1)f+2} \frac{\mathrm{Log}(n)}{n^\sigma}.$$

Since $(u_k)_{k\geq 0}$ and $(v_k)_{k\geq 0}$ are decreasing sequences converging towards 0, and since $u_{k+1} \leq v_k$, $k \geq 0$, we get

$$\begin{aligned} |L'(\sigma, \chi)| &\leq \frac{\mathrm{Log}(2)}{2^\sigma} + u_0 - (v_0 - u_1) - (v_1 - u_2) + \cdots \\ &\leq \frac{\mathrm{Log}(2)}{2^\sigma} + u_0 = \sum_{n=2}^{N+2} \frac{\mathrm{Log}(n)}{n^\sigma}. \quad \square \end{aligned}$$

Theorem 3 (see [12, Lemma 11.10]). *Let χ be a primitive quadratic character of conductor f. Then*

$$L(\sigma, \chi) \geq 0 \quad \textit{for} \begin{cases} \sigma \geq \sigma_0 = 1 - \frac{2}{\sqrt{f}\,\mathrm{Log}(f)} & \textit{if}\ \chi(-1) = +1, \\ \sigma \geq \sigma_0 = 1 - \frac{2\pi}{\sqrt{f}\,\mathrm{Log}^2(f)} & \textit{if}\ \chi(-1) = -1. \end{cases}$$

Hence, $L(\sigma, \chi) \geq 0$ *for* $\sigma \geq \sigma_1 = 1 - 2/(f-2)\,\mathrm{Log}(f)$.

Proof. Since $\sigma \mapsto L(\sigma, \chi)$ has no real zero in the open interval $]0, 1[$ for $f \leq 24$ (see [9]), we may assume that we have $f \geq 24$. Let us first assume that χ is even, and let $\mathbf{k}_2$ be the real quadratic field with conductor f. Then $L(1, \chi) = 2h\,\mathrm{Log}(\varepsilon_0)/\sqrt{f} \geq \mathrm{Log}(f-4)/\sqrt{f}$ where $\varepsilon_0 \geq (\sqrt{f-4} + \sqrt{f})/2 \geq \sqrt{f-4}$ is the fundamental unit of $\mathbf{k}_2$ and where $h \geq 1$ is the class number of $\mathbf{k}_2$. Let σ be such that $\sigma_0 \leq \sigma \leq 1$. Then $L(\sigma, \chi) \geq 0$. Indeed, if we had $L(\sigma, \chi) < 0$, then from Lemma (iii) above and since we have $N \leq (f-1)/2$, we would get a contradiction from

$$\begin{aligned} \frac{\mathrm{Log}(f-4)}{\sqrt{f}} &\leq L(1, \chi) < L(1, \chi) - L(\sigma, \chi) \\ &\leq (1-\sigma) \max_{\sigma_0 \leq \sigma \leq 1} L'(\sigma, \chi) \\ &\leq (1-\sigma_0) \exp\left(\frac{2\,\mathrm{Log}((f+3)/2)}{\sqrt{f}\log(f)}\right) \sum_{2 \leq n \leq (f+3)/2} \frac{\mathrm{Log}(n)}{n} \\ &\leq (1-\sigma_0)\frac{1}{2}\mathrm{Log}(f-4)\,\mathrm{Log}(f) = \frac{\mathrm{Log}(f-4)}{\sqrt{f}} \end{aligned}$$

where the last inequality is valid for $f \geq 24$.

In the same way, we get the desired result if χ is odd using $L(1, \chi) \geq \pi/\sqrt{f}$, $f \geq 5$.

The third result follows from the first and second ones. $\square$

Corollary c (see [12, Corollary 11.17]). *Let p be an odd prime. Then we have the following lower bound for the relative class number $h^*(\mathbf{K})$ of the cyclotomic field $\mathbf{K} = \mathbf{Q}(\zeta_{p^a})$, $a \geq 1$, of degree $2N = [\mathbf{K}\colon \mathbf{Q}] = \phi(p^a)$:*

$$h^*(\mathbf{K}) \geq \frac{1}{76}\left(\frac{N}{39}\right)^{N/2} \frac{1}{\mathrm{Log}(2N)},$$

so that $2N = \phi(p^a) \geq 100$ *implies* $h^*(\mathbf{K}) > 1$. *Moreover,* $p \geq 89$ *implies* $h^*(\mathbf{K}) > 1$.

Proof. Set $h(p) = (1 - 1/p)p^{1/(p-1)}$, so that we have $h(5) \geq h(p) \geq 1$. We note that

$$d(\mathbf{K}) = (2N/h(p))^{2N} \leq (2N)^{2N}, \qquad w(\mathbf{K}) = 2p^a \geq 2N,$$
$$d(\mathbf{K})/d(\mathbf{k}) = \sqrt{pd(\mathbf{K})} = \sqrt{p}(2N/h(p))^N,$$
$$\mathrm{Res}_1(\zeta_{\mathbf{k}}) \leq (\pi^2/6)^{(N-1)/2} \quad \text{[thanks to Lemma (ii)].}$$

Noticing that we have $d(\mathbf{K}) \geq p^{p-2}$, then thanks to Theorem 3 we may apply Theorem 2(b), so that we get the following lower bound from which we get the desired results:

$$h^*(\mathbf{K}) \geq \frac{2\pi\sqrt{6}}{15e} p^{1/4} e^{-\pi h(p)} \left(\frac{3N}{\pi^4 h(p)}\right)^{N/2} \frac{1}{\mathrm{Log}(2N)}. \quad \square$$

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