PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 2, February 1994

SIGN PROPERTIES OF GREEN'S FUNCTIONS FOR A FAMILY OF TWO-POINT BOUNDARY VALUE PROBLEMS

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(Communicated by Hal L. Smith)

ABSTRACT. Sign properties of and comparison theorems for Green's functions of a family of two-point boundary value problems for a general ordinary differential operator are obtained. The results extend known results for a two-term ordinary differential operator.

1. INTRODUCTION

Let $n > 1$ be an integer, and let $B > 0$ be given. Let $a_l \in C[0, B]$, $l = 1, \dots, n$, and define the linear differential operator L by

$$Ly = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y, \qquad 0 \le x \le B. \tag{1.1}$$

Let W denote the set of nonnegative integers, and, for each $k \in \{1, \dots, n-1\}$, let $\Omega_{n-k} \subset W^{n-k}$ be defined by

$$\Omega_{n-k} = \{\alpha = (\alpha_1, \dots, \alpha_{n-k}) \colon 0 \le \alpha_1 < \cdots < \alpha_{n-k} \le n-1\}. \tag{1.2(k)}$$

For each $\alpha \in \Omega_{n-k}$, $b \in (0, B]$, we shall consider homogeneous, two-point boundary conditions of the form

$$\begin{aligned} y^{(l)}(0) &= 0, \qquad l = 0, \dots, k-1; \\ y^{(l)}(b) &= 0, \qquad l = \alpha_1, \dots, \alpha_{n-k}. \end{aligned} \tag{1.3(k, \alpha, b)}$$

We shall denote the boundary conditions $(1.3(k, \alpha, b))$ by $T(k, \alpha, b)y = 0$. Note that, if $\alpha = (0, \dots, n-k-1)$, then $T(k, \alpha, b)y = 0$ represents two-point, $(k, n-k)$ conjugate boundary conditions. If $\alpha = (k, \dots, n-1)$, then $T(k, \alpha, b)y = 0$ represents two-point, $(k, n-k)$ right focal boundary conditions. Thus, the boundary conditions $(1.3(k, \alpha, b))$ represent a family of boundary conditions that are between conjugate and right focal type boundary conditions. See [7, 8, 11, 12, 19].

Recall [11] that L, defined by (1.1), is *right disfocal* on $[0, B]$ if the only solution of $Ly = 0$ satisfying $y^{(l)}(x_l) = 0$, $l = 0, \dots, n-1$, where $0 \le x_0 \le \cdots \le x_{n-1} \le B$, is $y \equiv 0$. Throughout §2, we shall assume that L is right disfocal on $[0, B]$.

Received by the editors May 8, 1992; presented to the American Mathematical Society on November 1, 1993 in Dayton, Ohio.

1991 *Mathematics Subject Classification*. Primary 34B27.

This research was performed while the first author was visiting Utah State University.

For each $k \in \{1, \ldots, n-1\}$, $\alpha \in \Omega_{n-k}$, $b \in (0, B]$, let $G(k, \alpha, b; x, s)$ denote the Green's function of the boundary value problem (BVP), $Ly = 0$, $0 \le x \le b$, $T(k, \alpha, b)y = 0$. Under the assumption that L is right disfocal on $[0, B]$, each $G(k, \alpha, b; x, s)$ exists. For each $l = 0, \ldots, n$, let $G_l(k, \alpha, b; x, s) = (\partial^l/\partial x^l)G(k, \alpha, b; x, s)$.

Define a partial order on Ω_{n-k} as follows: for $\alpha = (\alpha_1, \ldots, \alpha_{n-k})$, $\beta = (\beta_1, \ldots, \beta_{n-k}) \in \Omega_{n-k}$, we say that $\alpha \le \beta$ if and only if $\alpha_l \le \beta_l$, $l = 1, \ldots, n-k$. Moreover, $\alpha < \beta$ if $\alpha \le \beta$ and $\alpha \ne \beta$.

The purpose of this paper is to obtain comparison theorems for the family of Green's functions $G(k, \alpha, b; x, s)$. For example, we shall show that, if $\alpha, \beta \in \Omega_{n-k}$, $\alpha < \beta$, and $0 < b < B$, then

$$\begin{aligned}(1.4)\qquad (-1)^{n-k}G_l(k, \beta, b; x, s) &> (-1)^{n-k}G_l(k, \alpha, b; x, s)\\ &> 0, \qquad (x, s) \in (0, b) \times (0, b),\end{aligned}$$

$l = 0, \ldots, \alpha_1$. Moreover, we shall show that $G_i(k, \alpha, b; b, s) \ne 0$, $s \in (0, b)$, for $i \in \{0, \ldots, n-1\} \backslash \{\alpha_1, \ldots, \alpha_{n-k}\}$, and we shall determine its sign. Inequalities such as (1.4) play an important role in applications of monotone methods and cone theoretic methods to nonlinear BVP's; information concerning the sign of $G_i(k, \alpha, b; b, s)$ can play a crucial role in the construction of cones with nonempty interior. See, for example, [6–8].

Sign properties of Green's functions have been of interest to many authors. See, for example, [1, 3–10, 14–19, 21, 22]. The results of this paper are specifically motivated by the results of Peterson [17, 18], Elias [4], and Peterson and Ridenhour [19], who all consider the two-term operator $Ly = y^{(n)} + q(x)y$. Peterson [17, 18] considers the cases $q(x) < 0$ and $q(x) > 0$ independently and obtains inequalities similar to (1.4) for $l \le \min\{k-1, \alpha_1\}$. Elias [4], who points out that this bound on l is best possible, considers only the case $(-1)^{n-k}q(x) < 0$ and obtains inequalities similar to (1.4) for $l = 0, \ldots, n-1$ and partial derivatives of G with respect to s. Peterson and Ridenhour [19] handle the case where q may change sign, and again, the bound on l is best possible. Thus, the primary contribution of this paper is that inequalities similar to (1.4), which are already known for the two-term right disfocal differential operator, are obtained for the general right disfocal operator.

In §2 we shall obtain (1.4) and related inequalities for the family of boundary conditions $(1.3(k, \alpha, b))$. The proofs employ the observation that as a function of x, the difference, $G(k_2, \beta, b_2; x, s) - G(k_1, \alpha, b_1; x, s)$, is n times differentiable on $[0, \min\{b_1, b_2\}]$ and, hence, is a solution of $Ly = 0$, $0 \le x \le \min\{b_1, b_2\}$, or that, as a function of x, $(\partial/\partial b)G(k, \alpha, b; x, s)$ is n times differentiable on $[0, b]$ and, hence, a solution of $Ly = 0$, $0 \le x \le b$. The proofs then employ a double induction on k and $(\alpha_1 + \cdots + \alpha_{n-k})$. The induction on k replaces an adjoint argument employed by Peterson [17, 18] and Peterson and Ridenhour [19] for the two-term differential operator.

In §3 we shall outline how the results of §2 can be extended to a larger family of two-point boundary conditions. Note that in $(1.3(k, \alpha, b))$ the boundary conditions are stacked at the left end point, beginning with the functional value, and the boundary conditions are allowed to fan out at the right end point. In §3, assuming that L is left disfocal [11], the boundary conditions will first be stacked at the right end point and allowed to fan out at the left end point; analogues of the results of §2 will be obtained with a simple change of variable argument. Then, assuming that L is disfocal [13], the boundary conditions will be allowed to fan out at both end points and analogues of the results of §2 will be stated. For example, suppose L is fourth order and disfocal on $[0, b]$. Let $H(x, s)$ be the Green's function for the BVP, $Ly = 0$, $y(0) = y''(0) = 0$, $y(b) = y''(b) = 0$, and let $G(x, s)$ be the Green's function for the BVP, $Ly = 0$, $y(0) = y'(0) = 0$, $y(b) = y'(b) = 0$. Then the techniques employed in this paper can be used to show that $H(x, s) > G(x, s) > 0$, $(x, s) \in (0, b)\times(0, b)$. This information can then be employed when studying the elastic beam problem [20, pp. 175–179]. For example, the inequalities shown here give that a clamped beam is stiffer than a simply supported beam.

2. The case L right disfocal

Throughout this section, we shall assume that L, given by (1.1), is right disfocal on $[0, B]$. The purpose of this section is to obtain the following two theorems:

Theorem 2.1. *Let $k \in \{1, \dots, n-1\}$, $\alpha, \beta \in \Omega_{n-k}$, $\alpha < \beta$, and $0 < b \le B$. For $l = 0, \dots, \alpha_1$,*

$$(2.1)\qquad \begin{aligned}(-1)^{n-k}G_l(k, \beta, b; x, s) &> (-1)^{n-k}G_l(k, \alpha, b; x, s)\\ &> 0, \qquad (x, s) \in (0, b) \times (0, b),\end{aligned}$$

(2.2)
$$(-1)^{n-k}G_k(k, \beta, b; 0, s) > (-1)^{n-k}G_k(k, \alpha, b; 0, s) > 0, \qquad s \in (0, b).$$

Theorem 2.2. *Let $k \in \{1, \dots, n-1\}, \alpha, \beta \in \Omega_{n-k}$, and $\alpha_{n-k} < n-1$. Assume $\alpha \le \beta$, $0 < b_1 \le b_2 \le B$, and that one of the inequalities ($\alpha \le \beta$ or $b_1 \le b_2$) is strict. Then, for $l = 0, \dots, \alpha_1$,*

$$(2.3)\qquad \begin{aligned}(-1)^{n-k}G_l(k, \beta, b_2; x, s) &> (-1)^{n-k}G_l(k, \alpha, b_1; x, s)\\ &> 0, \qquad (x, s) \in (0, b_1) \times (0, b_1),\end{aligned}$$

$$(2.4)\qquad \begin{aligned}(-1)^{n-k}G_k(k, \beta, b_2; 0, s) &> (-1)^{n-k}G_k(k, \alpha, b_1; 0, s)\\ &> 0, \qquad s \in (0, b_1).\end{aligned}$$

Before we prove Theorems 2.1 and 2.2, we shall provide three technical lemmas. Lemma 2.3 has been obtained by Peterson and Ridenhour [19], and Lemma 2.5 has essentially been obtained by Eloe and Henderson [8].

Lemma 2.3. *Assume that L is right disfocal on $[0, B]$. Let $k \in \{1, \dots, n-1\}$ and $0 < b \le B$. Let $\alpha_1, \dots, \alpha_{n-k-1}$ be integers satisfying $0 \le \alpha_1 < \cdots < \alpha_{n-k-1} \le n-1$. Suppose $y(x)$ is a nontrivial solution of $Ly = 0$, $0 \le x \le b$, satisfying the $n-1$ boundary conditions*

$$y^{(l)}(0) = 0, \quad l = 0, \dots, k-1; \qquad y^{(l)}(b) = 0, \quad l = \alpha_1, \dots, \alpha_{n-k-1}.$$

Set $M = \max\{l\colon l \in \{0, \dots, n-1\},\ y^{(l)}(b) \ne 0\}$. Then $M \ge k$, and

(i) $y^{(l)}(b) \ne 0$, $l \in \{0, \dots, n-1\}\backslash\{\alpha_1, \dots, \alpha_{n-k-1}\}$,
(ii) $y^{(l)}(x) \ne 0$, $0 < x < b$ *and* $l \in \{0, \dots, \min\{k, \alpha_1\}\}$.

Further, there is an $\varepsilon > 0$ such that

(iii) *if* $l \in \{0, \dots, M-1\}$ *and* $y^{(l)}(b) = 0$, *then* $y^{(l)}(x)y^{(l+1)}(x) < 0$, $b - \varepsilon < x < b$,

(iv) *if* $l \in \{0, \dots, M-1\}$ *and* $y^{(l)}(b) \neq 0$, *then* $y^{(l)}(x)y^{(l+1)}(x) > 0$, $b - \varepsilon < x < b$.

Before we state and prove the next lemma, we introduce further notation. For $\alpha \in \Omega_{n-k}$, let $S(\alpha) = \alpha_1 + \cdots + \alpha_{n-k}$ denote the sum of the components of α. Then, for all $\alpha \in \Omega_{n-k}$, $(n-k)(n-k-1)/2 \leq S(\alpha) \leq (n-k)(n+k-1)/2$. Also, for $\alpha \in \Omega_{n-k}$, $i \in \{0, \dots, n-1\}$, we let $n(\alpha, i)$ denote the number of components of α which exceed i. In particular, $n(\alpha, i)$ counts the number of derivatives of y of higher order than i which the boundary conditions, $T(k, \alpha, b)y = 0$, specify to be zero at b.

Lemma 2.4. *Let* $k \in \{1, \dots, n-1\}$, $\alpha \in \Omega_{n-k}$, *and* $0 < b \leq B$. *Assume* $i \in \{0, \dots, n-1\} \backslash \{\alpha_1, \dots, \alpha_{n-k}\}$. *Then*

$$(-1)^{n(\alpha, i)} G_i(k, \alpha, b; b, s) > 0, \qquad s \in (0, b).$$

Proof. The proof is by induction on k. Let $k = 1$. First, consider $G(1, \alpha, b; x, s)$, where $\alpha = (0, 1, \dots, n-2)$ (the Green's function for the $(1, n-1)$ conjugate problem). By Levin's Theorem on signs of conjugate point Green's functions (see, e.g., [3] or [10]), $G_{n-1}(1, \alpha, b; b, s) > 0$, $s \in (0, b)$; in particular, the lemma holds for $k = 1$, $\alpha = (0, \dots, n-2)$, since $n(\alpha, n-1) = 0$. Now, let $\beta = (\beta_1, \dots, \beta_{n-1}) \in \Omega_{n-1}$, $\beta \neq \alpha$. The components of β exclude precisely one of the integers $0, \dots, n-2$. We denote that excluded integer by i. Then $n(\beta, i) = n - 1 - i$. Fix $s \in (0, b)$, and consider $g(x) = G(1, \beta, b; x, s) - G(1, \alpha, b; x, s)$. As noted in [22], g is n times continuously differentiable and solves $Ly = 0$ on $[0, b]$. Furthermore, g satisfies the boundary conditions

$$g(0) = 0; \qquad g^{(q)}(b) = 0, \quad q \in \{0, \dots, n-2\} \backslash \{i\};$$
$$g^{(n-1)}(b) = -G_{n-1}(1, \alpha, b; b, s) < 0.$$

Applying (i), (iii), and (iv) of Lemma 2.3, we see on counting down from the $(n-1)$st derivative of g that, in a sufficiently small left neighborhood of b, there are $n-2-i$ sign changes in the finite sequence, $g^{(n-1)}(x)$, $g^{(n-2)}(x)$, $\dots$, $g^{(i)}(x)$. Therefore,

$$(-1)^{n-1-i} g^{(i)}(b) = (-1)^{n(\beta, i)} G_i(1, \beta, b; b, s) > 0.$$

This proves the lemma in the case $k = 1$.

Next, we suppose Lemma 2.4 holds for $k - 1 \in \{1, \dots, n-2\}$ and establish the truth for k. The proof here is by induction on $S(\alpha)$. To begin, we establish the lemma when $S(\alpha)$ is minimized over Ω_{n-k}; that is, when $\alpha = (0, \dots, n-k-1)$ and $S(\alpha) = (n-k)(n-k-1)/2$. By Levin's Theorem, $G_{n-k}(k, \alpha, b; b, s) > 0$, $s \in (0, b)$, establishing the lemma for $i = n - k$, since $n(\alpha, n-k) = 0$. Take any i with $n - k < i \leq n - 1$, and let $\beta = (0, \dots, n-k-1, i)$. Note that $\beta \in \Omega_{n-k+1}$ and $n(\beta, n-k) = 1$. Thus, by the inductive assumption on $k - 1$, we know $G_{n-k}(k-1, \beta, b; b, s) < 0$, $s \in (0, b)$. For fixed $s \in (0, b)$, let $g(x) = G(k, \alpha, b; x, s) - G(k-1, \beta, b; x, s)$. Then $Lg = 0$ on $[0, b]$ with

$$g^{(j)}(0) = 0, \quad j = 0, \dots, k-2; \qquad g^{(j)}(b) = 0, \quad j = 0, \dots, n-k-1;$$
$$g^{(n-k)}(b) = G_{n-k}(k, \alpha, b; b, s) - G_{n-k}(k-1, \beta, b; b, s) > 0.$$

By Lemma 2.3, $g^{(i)}(b) = G_i(k, \alpha, b; b, s) > 0$, $s \in (0, b)$. This establishes the lemma for all $\alpha \in \Omega_{n-k}$ with $S(\alpha) = (n-k)(n-k-1)/2$.

Our inductive assumption on $S(\alpha)$ is that Lemma 2.4 holds for all $\alpha \in \Omega_{n-k}$ with $(n-k)(n-k-1)/2 \le S(\alpha) < m$, where m is an integer with $(n-k)(n-k-1)/2 < m \le (n-k)(n+k-1)/2$. The proof will be complete once we establish the truth of the lemma for all $\alpha \in \Omega_{n-k}$ with $S(\alpha) = m$. Let such an α be given. Take $i \in \{0, \dots, n-1\} \backslash \{\alpha_1, \dots, \alpha_{n-k}\}$. Consider first the case where $i < \alpha_{n-k}$. Choose the minimum number $j \in \{1, \dots, n-k\}$ such that $i < \alpha_j$. Let $\beta = (\beta_1, \dots, \beta_{n-k}) \in \Omega_{n-k}$ be such that $\beta_p = \alpha_p$ for $p \ne j$ and $\beta_j = i$. Since $S(\beta) < S(\alpha) = m$, we know the signs of the derivatives of $G(k, \beta, b; x, s)$ at $x = b$. Note that $n(\alpha, i) = n(\alpha, \alpha_j) + 1 = n(\beta, \alpha_j) + 1$. For fixed $s \in (0, b)$, let $g(x) = G(k, \alpha, b; x, s) - G(k, \beta, b; x, s)$. Then $Lg = 0$ on $[0, b]$ with

$$\begin{aligned} &g^{(q)}(0) = 0, && q = 0, \dots, k-1; \\ &g^{(q)}(b) = 0, && q \in \{\alpha_1, \dots, \alpha_{n-k}\} \backslash \{\alpha_j\}; \\ &g^{(\alpha_j)}(b) = -G_{\alpha_j}(k, \beta, b; b, s), && \end{aligned}$$

since $G_{\alpha_j}(k, \alpha, b; b, s) = 0$. By the inductive assumption, the sign of $g^{(\alpha_j)}(b)$ is the opposite of the sign of $(-1)^{n(\beta, \alpha_j)}$. We then apply Lemma 2.3 and see that $(-1)^{n(\beta, \alpha_j)+1} g^{(i)}(b) > 0$, since $g^{(q)}(b) \ne 0$ for $q \in \{\alpha_1, \dots, \alpha_{n-k}) \backslash \{\alpha_j\}$. Hence, $G_i(k, \alpha, b; b, s) = g^{(i)}(b)$ has the same sign as $(-1)^{n(\alpha, i)}$, which is the desired conclusion.

As a last step, we consider the case $i > \alpha_{n-k}$. Let $\beta = (\alpha_1, \dots, \alpha_{n-k}, i)$. Then $\beta \in \Omega_{n-k+1}$, and the signs of the derivatives at b of $G(k-1, \beta, b; x, s)$ are known because of the inductive assumption on k. For fixed $s \in (0, b)$, let $g(x) = G(k, \alpha, b; x, s) - G(k-1, \beta, b; x, s)$. Then $Lg = 0$ on $[0, b]$ with

$$g^{(q)}(0) = 0, \quad q = 0, \dots, k-2; \qquad g^{(q)}(b) = 0, \quad q = \alpha_1, \dots, \alpha_{n-k}; \\ g^{(i)}(b) = G_i(k, \alpha, b; b, s).$$

Since $m > (n-k)(n-k-1)/2$, we choose $j \in \{0, \dots, n-k-1\} \backslash \{\alpha_1, \dots, \alpha_{n-k}\}$. Since $j < \alpha_{n-k}$, we know from above that the sign of $G_j(k, \alpha, b; b, s)$ agrees with that of $(-1)^{n(\alpha, j)}$. We also see that the sign of $G_j(k-1, \beta, b; b, s)$ is $(-1)^{n(\alpha, j)+1}$ since $n(\beta, j) = n(\alpha, j) + 1$. Then

$$g^{(j)}(b) = G_j(k, \alpha, b; b, s) - G_j(k-1, \beta, b; b, s),$$

so $\operatorname{sgn}[g^{(j)}(b)] = (-1)^{n(\alpha, j)}$. The finite sequence $g^{(j)}(b), \dots, g^{(i)}(b)$ has $n(\alpha, j)$ sign changes by Lemma 2.3. Hence, $g^{(i)}(b) = G_i(k, \alpha, b; b, s) > 0$ as desired. This completes the proof of Lemma 2.4.

Let $H(k, \alpha, b; x, s) = (\partial/\partial b) G(k, \alpha, b; x, s)$. Eloe and Henderson [8] (or see [1, 6]) have proved the following lemma and corollary which give that, under the condition that $\alpha_{n-k} < n-1$, $(-1)^{n-k} H_{\alpha_1}$ is strictly positive. In particular, the following lemma gives that $(-1)^{n-k} G_{\alpha_1}$ is monotone increasing as a function of b.

Lemma 2.5. *Let $k \in \{1, \dots, n-1\}$, $\alpha \in \Omega_{n-k}$, $b \in (0, B]$, and $s \in (0, b)$. Then $H(k, \alpha, b; x, s)$, as a function of x, is the unique solution of the BVP,*

$$Ly = 0, \quad 0 \le x \le b; \qquad y^{(l)}(0) = 0, \quad l = 0, \dots, k-1;$$
$$y^{(l)}(b) = -G_{l+1}(k, \alpha, b; b, s), \qquad l = \alpha_1, \dots, \alpha_{n-k}.$$

Corollary 2.6. *Let $k \in \{1, \dots, n-1\}$ and $\alpha \in \Omega_{n-k}$. Assume $\alpha_{n-k} < n-1$. For $0 < b_1 < b_2 \le B$,*

$$(-1)^{n-k} G_{\alpha_1}(k, \alpha, b_2; x, s) > (-1)^{n-k} G_{\alpha_1}(k, \alpha, b_1; x, s), \qquad (x, s) \in (0, b_1) \times (0, b_1),$$

$$(-1)^{n-k} G_k(k, \alpha, b_2; 0, s) > (-1)^{n-k} G_k(k, \alpha, b_1; 0, s), \qquad s \in (0, b_1).$$

Proof. Since H satisfies the BVP given in Lemma 2.5 and the signs of $y^{(l)}$ are determined by Lemma 2.4, it follows that (details are provided in [8])

$$(-1)^{n-k} H_{\alpha_1}(k, \alpha, b; x, s) > 0, \qquad (x, s) \in (0, b) \times (0, b),$$
$$(-1)^{n-k} H_k(k, \alpha, b; 0, s) > 0, \qquad s \in (0, b).$$

Remark. Without further assumptions on the signs of the coefficients of (1.1), the condition $\alpha_{n-k} < n-1$ cannot be removed. Let $\alpha = (k, \dots, n-1)$, so that $(1.3(k, \alpha, b))$ represents $(k, n-k)$ right focal boundary conditions. Then $G(k, \alpha, b; x, s)$, the Green's function for the BVP, $y^{(n)} = 0$, $0 \le x \le b$, $T(k, \alpha, b)y = 0$, is independent of b (see [4, 14] or apply Lemma 2.5); i.e., $H \equiv 0$.

For another example, consider the Green's function $G(1, \alpha, b; x, s)$ of the BVP, $y'' - y = 0$, $y(0) = y'(b) = 0$. Here, H satisfies the BVP, $y'' - y = 0$, $y(0) = 0$, $y'(b) = -G_2(1, \alpha, b; b, s) = -G(1, \alpha, b; b, s) > 0$. Thus, $G(1, \alpha, b; x, s)$ is increasing as a function of b, whereas Corollary 2.6 gives that, if $\alpha_1 < 1$, $-G(1, \alpha, b; b, s)$ is increasing as a function of b. In general, if $\alpha_{n-k} = n-1$, then $H_{n-1}(k, \alpha, b; b, s) = -G_n(k, \alpha, b; b, s)$, and results concerning the sign of $H_{n-1}(k, \alpha, b; b, s)$ require sign conditions on the coefficients a_l, $l = 1, \dots, n$, in (1.1). For further discussion along these lines, refer to Nehari [13, 14] or Elias [4].

We are now in a position to prove Theorems 2.1 and 2.2. We shall first determine the sign of the Green's functions in Theorem 2.7 and then compare the Green's functions in Theorem 2.8.

Theorem 2.7. *Assume $\alpha_1 \le k-1$, and let $\alpha = (\alpha_1, \dots, \alpha_1 + n - k - 1)$. Then, for $0 < b \le B$,*

$$(-1)^{n-k} G_{\alpha_1}(k, \alpha, b; x, s) > 0, \qquad (x, s) \in (0, b) \times (0, b), \tag{2.5}$$

$$(-1)^{n-k} G_k(k, \alpha, b; 0, s) > 0, \qquad s \in (0, b). \tag{2.6}$$

Proof. We first argue that $G_{\alpha_1}(k, \alpha, b; x, s)$ or $G_k(k, \alpha, b; 0, s)$ do not change sign in $(0, b) \times (0, b)$ or $(0, b)$, respectively. The argument is analogous to that argument found in Coppel [3, pp. 106, 107]. Suppose that, for some $c \in (0, b)$, $G_{\alpha_1}(k, \alpha, b; c, s)$ changes sign. Find $f \in C[0, b]$ such that $f(x) > 0$, $0 < x < b$, and such that

$$\int_0^b G_{\alpha_1}(k, \alpha, b; c, s) f(s)\, ds = 0.$$

Then $h(x) = \int_0^b G(k, \alpha, b; x, s) f(s)\, ds$ satisfies $Ly = 0$, $0 \le x \le b$, and $h^{(l)}(0) = 0$, $l = 0, \dots, k-1$, $h^{(l)}(b) = 0$, $l = \alpha_1, \dots, \alpha_1 + n - k - 1$, and $h^{(\alpha_1)}(c) = 0$. Since $\alpha_1 < k$, Muldowney's Mean Value Theorem [12, Corollary 1, p. 375] applies and Lh changes sign in $(0, b)$. But $Lh = f$ so $G_{\alpha_1}(k, \alpha, b; c, s)$ does not change sign.

Now, suppose that, for some $s \in (0, b)$, $G_{\alpha_1}(k, \alpha, b; x, s)$ changes sign for some $x \in (0, b)$. It then follows from the preceding paragraph that there is $c \in (0, b)$ such that $G_{\alpha_1}(k, \alpha, b; c, s) = 0$, $0 < s < b$. Again, Muldowney's Mean Value Theorem can be applied to obtain a contradiction.

Thus, $G_{\alpha_1}(k, \alpha, b; x, s)$ does not change sign on $(0, b) \times (0, b)$. A similar argument gives that $G_k(k, \alpha, b; 0, s)$ does not change sign for $0 < s < b$. Moreover, since $\alpha = (\alpha_1, \dots, \alpha_1 + n - k - 1)$ and $G_{\alpha_1+n-k}(k, \alpha, b; b, s) > 0$ by Lemma 2.4, it follows by Taylor's Theorem that $(-1)^{n-k} G_{\alpha_1}(k, \alpha, b; x, s) \ge 0$, $(x, s) \in (0, b) \times (0, b)$, and that $(-1)^{n-k} G_k(k, \alpha, b; 0, s) \ge 0$, $s \in (0, b)$.

Finally, strict inequalities in (2.5) and (2.6) follow immediately from Corollary 2.6. Let $(x, s) \in (0, b) \times (0, b)$. Choose b_1 such that $\max\{x, s\} < b_1 < b$. Then

$$0 \le (-1)^{n-k} G_{\alpha_1}(k, \alpha, b_1; x, s) < (-1)^{n-k} G_{\alpha_1}(k, \alpha, b; x, s),$$
$$0 \le (-1)^{n-k} G_k(k, \alpha, b_1; 0, s) < (-1)^{n-k} G_k(k, \alpha, b; 0, s),$$

and the proof of Theorem 2.7 is complete.

Remark. An alternative proof of Theorem 2.7 can be constructed by: (i) extending Gustafson's convergence principle in [9] to cover Green's functions where the boundary conditions are as given by $T(k, \alpha, b)y = 0$; (ii) applying Lemma 2.4 to determine signs of appropriate derivatives of $G(k, \alpha, b; x, s)$ for x just to the left of b (with s fixed); and (iii) applying Corollary 2.6 together with Rolle's theorem and the convergence principle to show that the sign of G_{α_1} for x near b persists for $x \in (0, b)$. The arguments do not require Muldowney's Mean Value Theorem.

Theorem 2.8. *Let* $k \in \{1, \dots, n-1\}$, $\alpha, \beta \in \Omega_{n-k}$, $\alpha < \beta$, *and* $0 < b \le B$. *Then, for* $l = 0, \dots, \alpha_1$,

$$(2.7) \qquad (-1)^{n-k} G_l(k, \beta, b; x, s) > (-1)^{n-k} G_l(k, \alpha, b; x, s), \quad (x, s) \in (0, b) \times (0, b),$$

$$(2.8) \quad (-1)^{n-k} G_k(k, \beta, b; 0, s) > (-1)^{n-k} G_k(k, \alpha, b; 0, s), \qquad s \in (0, b).$$

Proof. Let $\alpha, \beta \in \Omega_{n-k}$, and first consider the case where there exists $j \in \{1, \dots, n-k\}$ such that $\alpha_p = \beta_p$, if $p \ne j$, and $\beta_j = \alpha_j + 1$. Note that $\alpha_1 \le k - 1$. Let $s \in (0, b)$, and set $g(x) = G(k, \beta, b; x, s) - G(k, \alpha, b; x, s)$. Then $Lg = 0$ on $[0, b]$ with

$$g^{(q)}(0) = 0, \quad q = 0, \dots, k-1; \qquad g^{(q)}(b) = 0, \quad q \in \{\beta_1, \dots, \beta_{n-k}\} \backslash \{\beta_j\};$$
$$g^{(\beta_j)}(b) = -G_{\beta_j}(k, \alpha, b; b, s).$$

By Lemma 2.4, $(-1)^{n-k-j} g^{(\beta_j)}(b) < 0$. By Lemma 2.3, $(-1)^{n-k} g^{(\alpha_1)}(x) > 0$, $0 < x < b$; in particular, (2.7) holds for $l = \alpha_1$. It follows from the boundary conditions $(1.3(k, \alpha, b))$ and integration that (2.7) holds for $l = 0, \ldots, \alpha_1$. The truth of (2.7) for arbitrary $\alpha < \beta$ follows from this special case.

Finally, $g^{(k)}(0) \neq 0$, by right disfocality. By (2.7), the boundary conditions $(1.3(k, \alpha, b))$, and Taylor's theorem, $(-1)^{n-k} g^{(k)}(0) > 0$; thus (2.8) holds, and the proof of Theorem 2.8 is complete.

Theorems 2.1 and 2.2 now follow from Corollary 2.6 and Theorems 2.7 and 2.8.

3. A BROADER FAMILY OF BOUNDARY CONDITIONS

Define L by

$$Ly = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y, \qquad A \leq x \leq B, \tag{3.1}$$

where $a_l \in C[A, B]$, $l = 1, \ldots, n$. L is *left disfocal* on $[A, B]$ [11] if the only solution of $Ly = 0$ satisfying $y^{(l)}(x_l) = 0$, $l = 0, \ldots, n-1$, where $A \leq x_{n-1} \leq \cdots \leq x_0 \leq B$, is $y \equiv 0$. For each $\alpha \in \Omega_k$ given by $(1.2(n-k))$, $a \in [A, B)$, consider two-point boundary conditions of the form

$$(3.2(k, \alpha, a)) \qquad \begin{aligned} y^{(l)}(a) &= 0, \qquad l = \alpha_1, \ldots, \alpha_k; \\ y^{(l)}(B) &= 0, \qquad l = 0, \ldots, n-k-1. \end{aligned}$$

The results of §2 have analogues for the BVP, $Ly = 0$, $a \leq x \leq B$, $(3.2(k, \alpha, a))$. The proofs can be obtained directly with analogous arguments, or the results can be obtained from §2 with the change of variable $t = -x$, $r = -s$. In particular, if $G(k, \alpha, a; x, s)$ is the Green's function for $Ly = 0$, $a \leq x \leq B$, $(3.2(k, \alpha, a))$, then $G(k, \alpha, a; x, s) = (-1)^n H(-x, -s)$, where $H(t, r)$ is the Green's function for the BVP

$$\widehat{L}y = y^{(n)} - a_1(t)y^{(n-1)} + \cdots + (-1)^n a_n(t)y, \qquad -B \leq t \leq -a,$$

with boundary conditions $(1.3(n-k, \alpha, -a))$. This statement is readily verified using the four properties that uniquely characterize the Green's function of a BVP. See [2, p. 192]. Moreover, L is left disfocal on $[A, B]$ if and only if $\widehat{L}$ is right disfocal on $[-B, -A]$. Thus the sign properties of G are obtained directly from the sign properties of H. We state the analogues of Theorems 2.1 and 2.2.

Theorem 3.1. *Let $k \in \{1, \ldots, n-1\}$, $\alpha, \beta \in \Omega_k$, $\alpha < \beta$, and $A \leq a < B$. Then, for $l = 0, \ldots, \alpha_1$,*

$$\begin{aligned} (-1)^{n-k+l} G_l(k, \beta, a; x, s) &> (-1)^{n-k+l} G_l(k, \alpha, a; x, s) \\ &> 0, \qquad (x, s) \in (a, B) \times (a, B), \end{aligned}$$

$$G_{n-k}(k, \beta, a; B, s) > G_{n-k}(k, \alpha, a; B, s) > 0, \qquad s \in (a, B).$$

Theorem 3.2. *Let $k \in \{1, \ldots, n-1\}$, $\alpha, \beta \in \Omega_k$, and $\alpha_k < n-1$. Assume $\alpha \leq \beta$, $A \leq a_2 \leq a_1 < B$, and that one of the inequalities ($\alpha \leq \beta$ or $a_2 \leq a_1$) is strict. For $l = 0, \ldots, \alpha_1$,*

$$\begin{aligned} (-1)^{n-k+l} G_l(k, \beta, a_2; x, s) &> (-1)^{n-k+l} G_l(k, \alpha, a_1; x, s) \\ &> 0, \qquad (x, s) \in (a_1, B) \times (a_1, B), \end{aligned}$$

$$G_{n-k}(k, \beta, a_2; B, s) > G_{n-k}(k, \alpha, a_1; B, s) > 0, \qquad s \in (a_1, B).$$

In closing, assume that L is disfocal [13] on $[A, B]$. That is, assume there is no nontrivial solution of $Ly = 0$ on $[A, B]$ satisfying $y^{(l)}(x_l) = 0$, $x_l \in [A, B]$, $j = 0, \ldots, n-1$. Let $k \in \{1, \ldots, n-1\}$, and define $\Omega \subseteq \Omega_k \times \Omega_{n-k}$ by

$$\Omega = \{(\hat{\alpha}, \alpha) \colon \hat{\alpha} \in \Omega_k,\ \alpha \in \Omega_{n-k},\ \text{and } \operatorname{card}\{\mu \colon \hat{\alpha}_\mu \leq l\} + \operatorname{card}\{\mu \colon \alpha_\mu \leq l\} \geq l,\quad l = 0, \ldots, n-1\}.$$

For $(\hat{\alpha}, \alpha), (\hat{\beta}, \beta) \in \Omega$, we say that $(\hat{\alpha}, \alpha) \leq (\hat{\beta}, \beta)$ if $\hat{\alpha} \leq \hat{\beta}$ in Ω_k and $\alpha \leq \beta$ in Ω_{n-k}. Let $A \leq a < b \leq B$, and consider boundary conditions of the form

$$(3.3(\hat{\alpha}, \alpha, a, b)) \qquad \begin{aligned} y^{(l)}(a) &= 0, \qquad l = \hat{\alpha}_1, \ldots, \hat{\alpha}_k; \\ y^{(l)}(b) &= 0, \qquad l = \alpha_1, \ldots, \alpha_{n-k}. \end{aligned}$$

Let $G(\hat{\alpha}, \alpha, a, b; x, s)$ denote the Green's function of the BVP, $Ly = 0$, $a \leq x \leq b$, $(3.3(\hat{\alpha}, \alpha, a, b))$. An analogue of Lemma 2.3 can be obtained, and the inductive techniques employed in §2 can be employed here to obtain analogues of the results of §2. We close by stating analogues of Theorems 2.1 and 2.2 for the family of BVPs, $Ly = 0$, $a \leq x \leq b$, $(3.3(\hat{\alpha}, \alpha, a, b))$.

Theorem 3.3. *Let* $(\hat{\alpha}, \alpha), (\hat{\beta}, \beta) \in \Omega$, $(\hat{\alpha}, \alpha) \leq (\hat{\beta}, \beta)$, $(\hat{\alpha}, \alpha) \neq (\hat{\beta}, \beta)$, *and* $A \leq a < b \leq B$.

(i) *If* $\hat{\alpha}_1 = 0$, *then, for* $l = 0, \ldots, \alpha_1$,

$$(-1)^{n-k} G_l(\hat{\beta}, \beta, a, b; x, s) > (-1)^{n-k} G_l(\hat{\alpha}, \alpha, a, b; x, s) > 0, \qquad (x, s) \in (a, b) \times (a, b).$$

(ii) *If* $\alpha_1 = 0$, *then, for* $l = 0, \ldots, \hat{\alpha}_1$.

$$(-1)^{n-k+l} G_l(\hat{\beta}, \beta, a, b; x, s) > (-1)^{n-k+l} G_l(\hat{\alpha}, \alpha, a, b; x, s) > 0, \qquad (x, s) \in (a, b) \times (a, b).$$

Theorem 3.4. *Assume that* $(\hat{\alpha}, \alpha) \leq (\hat{\beta}, \beta)$, $\max\{\hat{\alpha}_k, \alpha_{n-k}\} < n-1$, *and* $A \leq a_2 \leq a_1 < b_1 \leq b_2 \leq B$, *and assume one of the inequalities* $(\hat{\alpha}, \alpha) \leq (\hat{\beta}, \beta)$, $a_2 \leq a_1$, *or* $b_1 \leq b_2$, *is strict.*

(i) *If* $\hat{\alpha}_1 = 0$, *then, for* $l = 0, \ldots, \alpha_1$,

$$(-1)^{n-k} G_l(\hat{\beta}, \beta, a_2, b_2; x, s) > (-1)^{n-k} G_l(\hat{\alpha}, \alpha, a_1, b_1; x, s) > 0, \qquad (x, s) \in (a_1, b_1) \times (a_1, b_1).$$

(ii) *If* $\alpha_1 = 0$, *then, for* $l = 0, \ldots, \hat{\alpha}_1$,

$$(-1)^{n-k+l} G_l(\hat{\beta}, \beta, a_2, b_2; x, s) > (-1)^{n-k+l} G_l(\hat{\alpha}, \alpha, a_1, b_1; x, s) > 0, \qquad (x, s) \in (a_1, b_1) \times (a_1, b_1).$$

Remark. Theorem 3.3 applies in the discussion, in the last paragraph of the introduction, of the elastic beam problem.

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