

SOME PRODUCTS OF β -ELEMENTS IN THE NOVIKOV E_2 TERM OF MOORE SPECTRA

JINKUN LIN

(Communicated by Frederick R. Cohen)

ABSTRACT. In this note, we prove trivialities and nontrivialities of products of some higher-order $\beta'_{tp^n/s}$ elements in the E_2 terms of the Adams-Novikov spectral sequence of Moore spectra.

1. INTRODUCTION

Let S be the sphere spectrum and M the Moore spectrum modulo a prime $p \geq 5$ given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S.$$

Consider the Brown-Peterson spectrum BP at p and the Adams-Novikov spectral sequence (ANSS) $\text{Ext}^{s,t} M = \text{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*M) \Rightarrow \pi_{t-s}M$. According to Miller and Wilson [1] and Miller, Ravenel, and Wilson [2], there are β -elements

$$\beta'_{tp^n/s} \in \text{Ext}^{1,*} M \quad \text{for} \quad \begin{cases} 1 \leq s \leq p^n + p^{n-1} - 1 & \text{if } p \nmid t \geq 2, \\ 1 \leq s \leq p^n & \text{if } t = 1, \end{cases}$$

such that their images under the boundary homomorphisms associated with the short exact sequence $0 \rightarrow BP_* \xrightarrow{p} BP_* \rightarrow BP_*/(p) \rightarrow 0$ are $\beta_{tp^n/s} \in \text{Ext}^{2,*} S = \text{Ext}_{BP_*BP}^{2,*}(BP_*, BP_*)$. We will write $\beta'_{tp^n/1}$ as β'_{tp^n} in $\text{Ext}^1 M$ and $\beta_{tp^n/1}$ as β_{tp^n} in $\text{Ext}^2 S$.

The present note gives some results on trivialities and nontrivialities of products of higher-order β -elements in $\text{Ext}^{*,*} M$, and we will consider p as a prime ≥ 5 throughout this note.

Theorem 1.1. *The following relations on products of β -elements in the E_2 terms of the ANSS of M hold:*

- (1) $\beta'_{mp^{n-1}/p^{n-1}} \beta_{tp^n/p^n} = 0$ in $\text{Ext}^3 M$ for $p \nmid t \geq 2$, $n \geq 2$, and $m = k(tp-1)$, $k \not\equiv 0, -1 \pmod{p}$.
- (2) $\beta'_{mp^{n-1}} \beta_{tp^n/a} = \beta'_{mp^{n-1}/p^{n-1}+1} \beta_{tp^n/p^{n-1}} = -\beta'_{mp^{n-1}/2} \beta'_{tp^n/a} \cdot h_0 \neq 0$ in $\text{Ext}^3 M$ for $p \nmid t \geq 2$, $n \geq 2$, and $m = k(tp-1)$, $k \not\equiv 0, -1 \pmod{p}$, where

Received by the editors June 20, 1991 and, in revised form, May 22, 1992.

1991 *Mathematics Subject Classification.* Primary 55T15.

Key words and phrases. Moore spectra, β -elements, Adams-Novikov spectral sequence.

$a = p^n + p^{n-1} - 1$ and $h_0 \in \text{Ext}^1 M$ is the v_1 -torsion free generator stated in [1, Theorem 1.1].

The triviality in (1) will be proved by using a result in [1], and the nontriviality in (2) will rely on the result on $\beta'_s \beta_{tp^n/a}$ in [7]. By using the results in Theorem 1.1, we have

Theorem 1.2. (1) The product $\beta'_{bp^{n-1}/p^{n-1}} \beta'_{tp^n/p^n} = 2t \cdot \beta'_{bp^{n-1}+(tp-1)p^{n-1}} \cdot h_0 \neq 0$ in $\text{Ext}^2 M$ for $p \nmid t \geq 2$, $n \geq 2$, and $b \not\equiv 0, 1 \pmod{p}$.

(2) $\beta'_{bp^n/s} \cdot h_0 \neq 0$ in $\text{Ext}^2 M$ for $2 \leq s \leq p^n + p^{n-1} - 1$, $n \geq 1$, $p \nmid b \geq 2$, and $b \not\equiv -1 \pmod{p}$ or $b = tp - 1$ with $p \nmid t \geq 2$.

(3) $\beta'_{bp^n} \cdot h_0 = 0$ in $\text{Ext}^2 M$ for $n \geq 1$ and $b = tp - 1$ with $p \nmid t \geq 2$.

2. PROOF OF THE MAIN THEOREMS

Let $\alpha: \Sigma^q M \rightarrow M$ be the Adams map and K_r be the cofibre of α^r given by the cofibration

$$(2.1) \quad \Sigma^{rq} M \xrightarrow{\alpha^r} M \xrightarrow{i'_r} K_r \xrightarrow{j'_r} \Sigma^{rq+1} M.$$

The cofibration (2.1) induces a short exact sequence $0 \rightarrow BP_*/(p) \xrightarrow{v'_r} BP_*/(p) \rightarrow BP_*/(p, v'_r) \rightarrow 0$ and then induces the Ext exact sequence

$$\dots \rightarrow \text{Ext}^{s,t} M \xrightarrow{v'_r} \text{Ext}^{s,t+rq} M \xrightarrow{(i'_r)_*} \text{Ext}^{s,t+rq} K_r \xrightarrow{(j'_r)_*} \text{Ext}^{s+1,t} M \rightarrow \dots$$

where we write $(j'_r)_*$ as the boundary homomorphism and $(i'_r)_*$ as the reduction.

From [3, p. 422], $i'_u j'_r: K_r \rightarrow \Sigma^{rq+1} K_u$ induces a cofibration

$$(2.2) \quad \Sigma^{rq} K_u \xrightarrow{\psi} K_{r+u} \xrightarrow{\rho} K_r \xrightarrow{i'_u j'_r} \Sigma^{rq+1} K_u,$$

which realizes the short exact sequence $0 \rightarrow BP_*/(p, v_1^u) \xrightarrow{v_1^r} BP_*/(p, v_1^{u+r}) \rightarrow BP_*(p, v_1^r) \rightarrow 0$ and induces the Ext exact sequence

$$\dots \rightarrow \text{Ext}^{s,t} K_u \xrightarrow{\psi_*} \text{Ext}^{s,t+rq} K_{u+r} \xrightarrow{\rho_*} \text{Ext}^{s,t+rq} K_r \xrightarrow{(i'_u j'_r)_*} \text{Ext}^{s+1,t} K_u \rightarrow \dots$$

where we write $(i'_u j'_r)_*$ as the boundary homomorphism and $\psi_* = v_1^r$. From the 3×3 lemma in the stable homotopy category, we can easily have

$$(2.3) \quad \psi i'_u = i'_{u+r} \alpha^r, \quad j'_r \rho = \alpha^u j'_{u+r}.$$

Note that the behavior of ψ_* , ρ_* , and $(i'_u j'_r)_*$ in the above Ext exact sequence is compatible with that of ψ , ρ , and $i'_u j'_r$ in the cofibration, i.e., we also have

$$(2.4) \quad \psi_*(i'_u)_* = (i'_{u+r})_* v_1^r, \quad (j'_r)_* \rho_* = v_1^u (j'_{u+r})_*$$

in the Ext stage.

If $r \equiv 0 \pmod{p}$, K_r is a split ring spectrum (cf. [5]), there exists $\bar{\delta} \in [\Sigma^{-1} K_r, K_r]$ such that

$$(2.5) \quad \bar{\delta} i'_r = i'_r \bar{\delta}, \quad j'_r \bar{\delta} = -\bar{\delta} j'_r, \quad (\bar{\delta} = ij)$$

and $\bar{\delta}$ is a derivation behaved on the products in $\pi_* K_r$, i.e.,

$$\bar{\delta} \bar{\delta} = 0, \quad \bar{\delta} \mu = \mu(\bar{\delta} \wedge 1_{K_r}) + \mu(1_{K_r} \wedge \bar{\delta}),$$

where $\mu: K_r \wedge K_r \rightarrow K_r$ is the associative and commutative multiplication of K_r . Hence $\bar{\delta}_*: \text{Ext}^{s,t} K_r \rightarrow \text{Ext}^{s+1,t} K_r$ also is a derivation behaved on the products in $\text{Ext}^{*,*} K_r$, i.e., $\bar{\delta}_* \bar{\delta}_* = 0$, and

$$(2.6) \quad \bar{\delta}_*(xy) = (\bar{\delta}_*x)y + (-1)^{|x|}x(\bar{\delta}_*y), \quad x, y \in \text{Ext}^{*,*} K_r.$$

Moreover, from (2.5) we have

$$(2.7) \quad \begin{aligned} \bar{\delta}_*(i'_r)_* &= (i'_r)_* \delta_*: \text{Ext}^{s,*} M \rightarrow \text{Ext}^{s+1,*} K_r, \\ (j'_r)_* \bar{\delta}_* &= -\delta_*(j'_r)_*: \text{Ext}^{s,*} K_r \rightarrow \text{Ext}^{s+2,*} M, \end{aligned}$$

where $\delta_*: \text{Ext}^{s,t} M \rightarrow \text{Ext}^{s+1,t} M$ is the boundary homomorphism induced by $\delta = ij \in [\Sigma^{-1}M, M]$, and it is similar that δ_* is a derivation behaved on the products in $\text{Ext}^{*,*} M$

$$(2.8) \quad \delta_*(xy) = (\delta_*x)y + (-1)^{|x|}x(\delta_*y), \quad x, y \in \text{Ext}^{*,*} M.$$

Proof of Theorem 1.1. (1) Briefly write $a = p^n + p^{n-1} - 1$. According to [1], $\beta'_{tp^n/a} = c_1(tp^n) \in \text{Ext}^1 M$ and $c_1(tp^n)$ is the v_1 -torsion generator of $\text{Ext}^1 M$ stated in [1, Theorem 1.1, p. 132]. Moreover, $\beta'_{tp^n/s} = v_1^{a-s} c_1(tp^n) = v_1^{a-s} \beta'_{tp^n/a}$, and $v_1^u \beta'_{tp^n/s+u} = \beta'_{tp^n/s}$ for $u+s \leq a$. Briefly write $d = p^{n-1}$. Then

$$(2.9) \quad \begin{aligned} (i'_d)_*(\beta'_{tp^n/p^n}) &= (i'_d)_* v_1^{d-1} \beta'_{tp^n/a} = \psi_* i'_*(\beta'_{tp^n/a}) \\ &= 2t \cdot \psi_*(v_2^{(tp-1)d} h_0) = 2t \cdot v_2^{(tp-1)d} \psi_*(h_0), \end{aligned}$$

where $\psi: \Sigma^{(d-1)q} K_1 \rightarrow K_d$ is the map in (2.2) and we use the relation $i'_* c_1(tp^n) = 2t \cdot v_2^{(tp-1)d} h_0$ in [1, Theorem 1.1(b)(iii)].

Since $h_0 \in \text{Ext}^{1,q} K_1$ converges to $i'ij\alpha i \in \pi_{q-1} K_1$ in the ANSS and $\bar{\delta} \psi i'ij\alpha i = \bar{\delta} i'_d \alpha^{d-1} i j \alpha i = i'_d i j \alpha^{d-1} i j \alpha i = 0 \in \pi_* K_d$, it follows that $\bar{\delta}_* \psi_*(h_0) = 0 \in \text{Ext}^2 K_d$. Hence, by applying the derivation $\bar{\delta}_*$ on (2.9), we have

$$(2.10) \quad \begin{aligned} (i'_d)_* \delta_*(\beta'_{tp^n/p^n}) &= \bar{\delta}_*(i'_d)_*(\beta'_{tp^n/p^n}) \\ &= 2t \cdot \bar{\delta}_*(v_2^{(tp-1)d} \psi_*(h_0)) = 2t \cdot \bar{\delta}_*(v_2^{(tp-1)d}) \cdot \psi_*(h_0). \end{aligned}$$

Since $v_2^{(tp-1)d} \in \text{Ext}^0 K_d$ by [1, Proposition 6.3] and $\bar{\delta}_*$ is a derivation (cf. (2.6)), then $\bar{\delta}_*(v_2^{(k+1)(tp-1)d}) = (k+1) \cdot v_2^{k(tp-1)d} \bar{\delta}_* v_2^{(tp-1)d}$, so by (2.10) we have

$$(2.11) \quad \begin{aligned} v_2^{k(tp-1)d} (i'_d)_* \delta_*(\beta'_{tp^n/p^n}) &= 2t \cdot v_2^{k(tp-1)d} \bar{\delta}_* v_2^{(tp-1)d} \cdot \psi_* h_0 \\ &= \frac{2t}{k+1} \bar{\delta}_*(v_2^{(k+1)(tp-1)d}) \cdot \psi_*(h_0) \\ &= \frac{2t}{k+1} \bar{\delta}_*(v_2^{(k+1)(tp-1)d} \psi_*(h_0)) \quad [\text{by (2.6) and } \bar{\delta}_* \psi_* h_0 = 0] \\ &= \frac{2t}{k+1} \bar{\delta}_* \psi_*(v_2^{(k+1)(tp-1)d} h_0) \quad [\text{since } \psi_* = v_1^{d-1}]. \end{aligned}$$

By applying the boundary homomorphism $(j'_d)_*: \text{Ext}^2 K_d \rightarrow \text{Ext}^3 M$, the left-hand side of (2.11) becomes $\beta'_{k(tp-1)d/d} \beta'_{tp^n/p^n}$ by the Yoneda product and the

right-hand side of (2.11) becomes

$$\begin{aligned} & \frac{2t}{k+1} \cdot (j'_d)_* \bar{\delta}_* \psi_*(v_2^{(k+1)(tp-1)d} h_0) \\ &= \frac{-2t}{k+1} \delta_* j'_*(v_2^{(k+1)(tp-1)d} h_0) \quad [\text{cf. (2.5), (2.4)}] \\ &= 0 \in \text{Ext}^3 M, \end{aligned}$$

since $j_* j'_*(v_2^{(k+1)(tp-1)d} h_0) = \beta_{(k+1)(tp-1)d} \alpha_1 \in \text{Ext}^3 S$ is divisible by p (cf. [2, Theorem 2.8(c), p. 477]). So we have the desired triviality.

(2) Let $m': M \wedge S \rightarrow M$ be the restriction of the multiplication $m: M \wedge M \rightarrow M$. Since $m' = m(1_M \wedge i)$, the following diagram commutes:

$$\begin{array}{ccc} \beta'_{mp^{n-1}} \otimes \beta_{tp^n/a} & \in \text{Ext}^1(BP_*, BP_* M) \otimes \text{Ext}^2(BP_*, BP_*) \\ \downarrow 1 \otimes i_* & \\ \beta'_{mp^{n-1}} \otimes \delta_* \beta'_{tp^n/a} & \in \text{Ext}^1(BP_*, BP_* M) \otimes \text{Ext}^2(BP_*, BP_* M) \\ \longrightarrow & \text{Ext}^3(BP_*, BP_* M \wedge S) \\ \downarrow (1_M \wedge i)_* & \\ \longrightarrow & \text{Ext}^3(BP_*, BP_* M \wedge M) \\ \xrightarrow{m'_*} & \text{Ext}^3(BP_*, BP_* M) \ni \beta'_{mp^{n-1}} \beta_{tp^n/a} \\ & \parallel \qquad \qquad \parallel \\ \xrightarrow{m_*} & \text{Ext}^3(BP_*, BP_* M) \ni \beta'_{mp^{n-1}} \delta_* \beta'_{tp^n/a} \end{array}$$

where the top and bottom rows are products in the ANSS. Hence, we have $\beta'_{mp^{n-1}} \beta_{tp^n/a} = \beta'_{mp^{n-1}} \delta_* \beta'_{tp^n/a}$ and by (2.8)

$$\begin{aligned} \beta'_{mp^{n-1}} \beta_{tp^n/a} &= \beta'_{mp^{n-1}} \delta_* \beta'_{tp^n/a} = \beta'_{mp^{n-1}/p^{n-1}+1} v_1^{p^{n-1}} \delta_* \beta'_{tp^n/a} \\ &= \beta'_{mp^{n-1}/p^{n-1}+1} \delta_*(v_1^{p^{n-1}} \beta'_{tp^n/a}) = \beta'_{mp^{n-1}/p^{n-1}+1} \beta_{tp^n/p^{n-1}} \end{aligned}$$

as desired. Moreover,

$$\begin{aligned} \beta'_{mp^{n-1}} \beta_{tp^n/a} &= \beta'_{mp^{n-1}/p^{n-1}+1} \delta_*(v_1 \beta'_{tp^n/p^n}) \\ &= \beta'_{mp^{n-1}/p^{n-1}+1} v_1 \delta_* \beta'_{tp^n/p^n} + \beta'_{mp^{n-1}/p^{n-1}+1} (\delta_* v_1) \beta'_{tp^n/p^n} \\ &= \beta'_{mp^{n-1}/p^{n-1}} \beta_{tp^n/p^n} - \beta'_{mp^{n-1}/p^{n-1}+1} \beta'_{tp^n/p^n} \cdot h_0 \\ &= -\beta'_{mp^{n-1}/2} \beta'_{tp^n/a} \cdot h_0 \quad [\text{the 1st term is zero from (1)}]. \end{aligned}$$

From [7, (4.1.3), p. 132] $\beta'_r \beta_{tp^n/a} \neq 0$ in $\text{Ext}^3 M$ if and only if $r \neq (up^e - p^{e-1}) - (tp^n - p^{n-1})$ for any $p \nmid u \geq 2$ and $e \geq 1$. Now if $mp^{n-1} = k(tp-1)p^{n-1} = (up^e - p^{e-1}) - (tp^n - p^{n-1})$ for some $p \nmid u \geq 2$ and $e \geq 1$, then $(k+1)(tp-1)p^{n-1} = (up-1)p^{e-1}$, and it is impossible since $k \not\equiv 0, -1 \pmod{p}$. Hence, $\beta'_{mp^{n-1}} \beta_{tp^n/a} \neq 0$. Q.E.D.

Proof of Theorem 1.2. (1) From (2.9) we have

$$\begin{aligned} v_2^{bd}(i'_d)_*(\beta'_{tp^n/p^n}) &= 2t \cdot v_2^{bd} \cdot v_2^{(tp-1)d} \psi_*(h_0) \\ &= 2t \cdot \psi_*(v_2^{(b+tp-1)d} h_0), \quad d = p^{n-1}. \end{aligned}$$

Hence, by applying the boundary homomorphism $(j'_d)_*: \text{Ext}^1 K_r \rightarrow \text{Ext}^2 M$, we have $\beta'_{bd/d} \beta'_{tp^n/p^n} = 2t \cdot j'_*(v_2^{(b+tp-1)d} h_0) = 2t \beta'_{(b+tp-1)d} h_0$. Moreover, if $b + tp - 1 \not\equiv 0, -1 \pmod{p}$, it follows from

$$0 \neq \beta_{(b+tp-1)d} \alpha_1 = j_*(\beta'_{(b+tp-1)d} h_0)$$

(cf. [2, Theorem 2.8(b)(i), p. 477]) that $\beta'_{(b+tp-1)d} h_0 \neq 0$.

(2) If $b \not\equiv -1 \pmod{p}$ and $\beta'_{bp^n/s} h_0 = 0$, then $\beta'_{bp^n} \cdot h_0 = 0$ in $\text{Ext}^2 M$, and so in $\text{Ext}^3 S$ we have

$$0 \neq \beta_{bp^n} \alpha_1 = j_*(\beta'_{bp^n} \cdot h_0) = 0,$$

which is a contradiction, where we use the result in [2, Theorem 2.8(b)(i), p. 477] on $\beta_{bp^n} \alpha_1 \neq 0$ in $\text{Ext}^3 S$ if $b \not\equiv -1 \pmod{p}$.

If $b = tp - 1$ with $p \nmid t \geq 2$, from Theorem 1.1(1) we have

$$\beta'_{bp^n/s} \beta'_{tp^{n+1}/p^{n+1}+p^n+1-s} \cdot h_0 \neq 0,$$

and so $\beta'_{bp^n/s} \cdot h_0 \neq 0$ in $\text{Ext}^2 M$.

(3) Let $b = tp - 1$ with $p \nmid t \geq 2$. It is known that there exists $f \in [\Sigma^* K_1, K_1]$ such that the induced BP_* homomorphism $f_* = v_2^{bp^n}$, and so $2t \cdot f' i' j \alpha_i \in \pi_* K_1$ is detected by $2t \cdot v_2^{bp^n} h_0 = i'_* c_1(tp^{n+1}) \in \text{Ext}^1 K_1$ (cf. [1, Theorem 1.1(b)(iii)]). Hence $j' f' i' j \alpha_i \in \pi_* M$ has BP filtration > 2 since $j'_* i'_* c_1(tp^{n+1}) = 0$. This means that $\beta'_{bp^n} \cdot h_0 = 0$ in $\text{Ext}^2 M$. Q.E.D.

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