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AN INTERPOLATION THEOREM AND A SHARP FORM OF A MULTILINEAR FRACTIONAL INTEGRATION THEOREM

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ABSTRACT. We prove a sharp interpolation theorem for Orlicz spaces with the Luxemburg norm. As a corollary we obtain a sharp form of an exponential integrability theorem, due to Grafakos, for the multilinear fractional integration operator. This generalizes a theorem of Adams.

1. INTRODUCTION

In his 1988 paper [A] Adams proved a sharp form of certain limiting cases of the Sobolev embedding theorem by first establishing the following exponential integrability theorem for the fractional integration operator I_α (Riesz potential of order α) defined by $I_\alpha(f)(x) = \int_{\mathbf{R}^n} f(x-y)|y|^{\alpha-n}dy$ $(0 < \alpha < n)$. The symbol $p' = \frac{p}{p-1}$ denotes the conjugate exponent to p, w_{n-1} the surface area of the unit sphere in $\mathbf{R}^n$, and $|\Omega|$ the Lebesgue measure of the set $\Omega \subset \mathbf{R}^n$.

Theorem A (Adams [A]). *For $p \in (1, \infty)$ and $\alpha = \frac{n}{p}$, there is a constant $c_0 = c_0(p)$ depending only on p such that for all $f \in L^p(\mathbf{R}^n)$ with support contained in a domain Ω in $\mathbf{R}^n$, $|\Omega| < \infty$,*

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\frac{n}{w_{n-1}}\left|\frac{I_\alpha(f)(x)}{\|f\|_p}\right|^{p'}\right)dx \le c_0. \tag{1}$$

Furthermore, (1) *fails if $\frac{n}{w_{n-1}}$ is replaced by a larger constant.*

Grafakos [G] extended this result to cover a multilinear analog of I_α, namely, the operator $I_\alpha(f_1, \dots, f_K)$ defined by

$$I_\alpha(f_1, \dots, f_K)(x) = \int_{\mathbf{R}^n} f_1(x-\theta_1 y)\cdots f_K(x-\theta_K y)|y|^{\alpha-n}\,dy,$$

where $\theta_j \in \mathbf{R}\backslash 0$, $1 \le j \le K$. He proved the following theorem, which may be called a multilinear fractional integration theorem.

Theorem B (Grafakos [G]). *Let $p \in (1, \infty)$, $\frac{1}{p} = \sum_{j=1}^K \frac{1}{p_j}$, $p_j \in (1, \infty]$, and $\alpha = \frac{n}{p}$. Assume the real numbers $\theta_j \neq 0$ are distinct. Let B be a ball in $\mathbf{R}^n$,*

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and let $f_j \in L^{p_j}$ *be supported in* B. *Then for any* $\gamma < 1$, *there exists a constant* $C_0(\gamma)$ *depending only on* n, α, θ_j*'s, and* γ *such that*

$$\frac{1}{|B|}\int_B \exp\left(\frac{n}{w_{n-1}}\gamma\left|\frac{LI_\alpha(f_1, \dots, f_K)(x)}{\|f_1\|_{p_1}\cdots\|f_K\|_{p_K}}\right|^{p'}\right)dx \le C_0(\gamma), \tag{2}$$

where $L = \prod_{j=1}^K |\theta_j|^{n/p_j}$. *Furthermore,* (2) *fails if* $\gamma > 1$.

Note that Theorem B leaves the case $\gamma = 1$ open. The proof of (2) was based on complicated estimates involving properties of the multisublinear maximal function defined by

$$M(f_1, \dots, f_K)(x) = \sup_{r>0}\frac{1}{|B(0, r)|}\int_{B(0,r)} |f_1(x-\theta_1 y)\cdots f_K(x-\theta_K y)|\,dy,$$

where $B(0, r)$ is the Euclidean ball of radius r centered at the origin in $\mathbf{R}^n$. (Here $\theta_j \in \mathbf{R}^n\backslash 0$ are assumed to be distinct.) The purpose of this note is to show that the end point case $\gamma = 1$ of (2) is actually an easy consequence of Theorem A, once we establish a sharp form of a multilinear interpolation theorem for Orlicz spaces (with the Luxemburg norm). We will state and prove this interpolation result in §2. Our observations may be combined with the known results and stated as follows:

Theorem 1. *Let* $K \ge 1$, $p \in (1, \infty)$, $\alpha = \frac{n}{p}$, $\frac{1}{p} = \sum_{j=1}^K \frac{1}{p_j}$, *and* $p_j \in (1, \infty]$. *Then there exists a constant* $c_0 = c_0(p)$ *depending only on* p *such that for all* $f_j \in L^{p_j}(\mathbf{R}^n)$ *with support contained in a domain* Ω *in* $\mathbf{R}^n$, $|\Omega| < \infty$,

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\frac{n}{w_{n-1}}\left|\frac{LI_\alpha(f_1, \dots, f_K)(x)}{\|f_1\|_{p_1}\cdots\|f_K\|_{p_K}}\right|^{p'}\right) dx \le c_0, \tag{3}$$

where $L = \prod_{j=1}^K |\theta_j|^{n/p_j}$. *Furthermore,* (3) *fails if* $\frac{n}{w_{n-1}}$ *is replaced by a larger constant.*

Remark. The constant c_0 in (3) is in fact identical to the one appearing in (1). So it depends only on the ratio $n/\alpha = p$. On the other hand, note that the constant $C_0(\gamma)$ in (2) depends on the other parameters also. In particular, $C_0(\gamma) \to \infty$, as $\gamma \to 1^-$.

Theorem 1 will be proved in §3.

2. A multilinear interpolation theorem for Orlicz spaces

Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space. A convex function $Q: [0, \infty) \to [0, \infty]$ is called a Young's function if $Q(0) = 0$ and Q is not identically 0 or ∞ on $(0, \infty)$. Given a measurable function f on X we let

$$\|f\|_Q = \inf\left\{l > 0 : \int Q\left(\frac{|f|}{l}\right)d\mu \le 1\right\} \quad \text{(Luxemburg norm)},$$

$$\|f\|_Q^* = \sup_{\|g\|_{Q^*}\le 1}\left|\int fg\,d\mu\right| \quad \text{(Orlicz norm)},$$

where Q^* denotes Young's complementary function of Q (defined by $Q^*(t) = \sup_{s\ge 0}[st - Q(s)]$ for $t \ge 0$). $\|f\|_Q$ and $\|f\|_Q^*$ are equivalent norms on the

Orlicz space $L^Q(X) = \{f : \|f\|_Q < \infty\}$. The letters Φ, Q, and R (with subscripts) will stand for Young's functions. Given a Young's function Q its (generalized) inverse is defined by $Q^{-1}(t) = \sup\{l \geq 0 : Q(l) < t\}$ (for $t > 0$). We will write R_ℓ for Young's complementary function of Q_ℓ, $\ell = 0, 1$. R_s will denote the intermediate function defined to be the inverse of the function $R_s^{-1}(t) = (R_0^{-1}(t))^{1-s}(R_1^{-1}(t))^s$ for $0 \leq s \leq 1$. For more details on Orlicz spaces see [KR, R].

We now state our interpolation result. Note that all the norms in the statement of Theorem 2 are the *Luxemburg* norms (see the remark after the proof of the theorem). Let $(X_j, \mathscr{M}_j, \mu_j)$ be σ-finite measure spaces and

$$T: L^{\Phi_{1,\ell}}(X_1) \times \cdots \times L^{\Phi_{K,\ell}}(X_K) \to L^{Q_\ell}(X)$$

denote a multilinear operator. M_ℓ will denote constants independent of the functions f_j.

Theorem 2. *Let T be a multilinear operator such that*

$$\|T(f_1, \dots, f_K)\|_{Q_\ell} \leq M_\ell \prod_{j=1}^{K} \|f_j\|_{\Phi_{j,\ell}}, \qquad \ell = 0, 1.$$

If $\Phi_{j,s}^{-1}(t) = (\Phi_{j,0}^{-1}(t))^{1-s} \cdot (\Phi_{j,1}^{-1}(t))^s$, $1 \leq j \leq K$, $0 \leq s \leq 1$, and $Q_s^+ = (R_s)^*$, then

$$\|T(f_1, \dots, f_K)\|_{Q_s^+} \leq M_0^{1-s} M_1^s \prod_{j=1}^{K} \|f_j\|_{\Phi_{j,s}}, \tag{4}$$

where f_j are (integrable) simple functions.

The following lemma is stated on p. 135 of [KR] in the case when μ is finite. Essentially the same proof can be used to show that it is valid when μ is σ-finite and $f \in L^1 \cap L^Q(\mu)$. We wish to thank Steve Bellenot for helpful conversations about the lemma.

Lemma 3. *If Q and Q^* are continuous and $f \in L^1 \cap L^Q(\mu)$ then*

$$\|f\|_Q = \sup\left\{\left|\int fg\, d\mu\right| : \|g\|_{Q^*}^* \leq 1\right\}.$$

Proof of Theorem 2. As was observed in [R] we may assume all of our Young's functions are continuous and strictly increasing. By Lemma 3 we have

$$\|T(f_1, \dots, f_K)\|_{Q_s^+} = \sup\left\{\left|\int T(f_1, \dots, f_K) g\, d\mu\right| : \|g\|_{R_s}^* \leq 1\right\},$$

if the f_j's are integrable simple functions. Fix a number $s \in (0, 1)$. For $1 \leq j \leq K$, $\ell = 0, 1$, and $z \in \mathbf{C}$, let f_j and g be (integrable) simple functions with $\|f_j\|_{\Phi_{j,s}} = 1$ and $\|g\|_{R_s}^* = 1$. Then it suffices to show that $|I| = |\int T(f_1, \dots, f_K) g\, d\mu| \leq M_0^{1-s} M_1^s$. Let $\alpha_{j,\ell} = \Phi_{j,\ell}^{-1}$, $\alpha_{j,z}(t) = (\alpha_{j,0}(t))^{1-z} \cdot (\alpha_{j,1}(t))^z$, $\beta_\ell = R_\ell^{-1}$, and $\beta_z(t) = (\beta_0(t))^{1-z} \cdot (\beta_1(t))^z$. Define

$$f_{j,z} = \alpha_{j,z}(\Phi_{j,s}(|f_j|)) \cdot e^{i \arg(f_j)}.$$

Let $\varepsilon > 0$. Since R_s is continuous by assumption, we have

$$1 = \|g\|_{R_s}^* = \inf_{k>0} \frac{1}{k}\left(1 + \int R_s(k|g|)d\mu\right)$$

(see [KR, p. 92]). Hence there exists a number $k_{s,\varepsilon} \in (0, \infty)$ such that

$$\frac{1}{k_{s,\varepsilon}}\left(1 + \int R_s(k_{s,\varepsilon}|g|)\right) \le 1 + \varepsilon.$$

Now define

$$g_z = \frac{1}{k_{s,\varepsilon}} \beta_z(R_s(k_{s,\varepsilon}|g|)) \cdot e^{i\arg(g)}.$$

Then $I(z) = \int T(f_{1,z}, \dots, f_{K,z}) g_z d\mu$ can be shown to be continuous in the strip $\{z \in \mathbf{C} : 0 \le \operatorname{Re} z \le 1\}$ and analytic in $\{z \in \mathbf{C} : 0 < \operatorname{Re} z < 1\}$ (see [R]). By Hölder's inequality (6),

$$\begin{aligned} |I(iy)| &\le \|T(f_{1,iy}, \dots, f_{K,iy})\|_{Q_0} \cdot \|g_{iy}\|_{R_0}^* \\ &\le M_0 \|g_{iy}\|_{R_0}^* \prod_{j=1}^{K} \|f_{j,iy}\|_{\Phi_{j,0}} \qquad (y \in \mathbf{R}). \end{aligned}$$

Now $\int \Phi_{j,0}(|f_{j,iy}|)d\mu_j = \int \Phi_{j,0}(\alpha_{j,0}(\Phi_{j,s}(|f_j|))) = \int \Phi_{j,s}(|f_j|) \le 1$, since $\|f_j\|_{\Phi_{j,s}} = 1$. So $\|f_{j,iy}\|_{\Phi_{j,0}} \le 1$. Also,

$$\begin{aligned} \|g_{iy}\|_{R_0}^* &= \inf_{k>0} \frac{1}{k}\left(1 + \int R_0(k|g_{iy}|)\right) \le \frac{1}{k_{s,\varepsilon}}\left(1 + \int R_0(k_{s,\varepsilon}|g_{iy}|)\right) \\ &= \frac{1}{k_{s,\varepsilon}}\left(1 + \int R_0(\beta_0(R_s(k_{s,\varepsilon}|g|)))\right) \\ &= \frac{1}{k_{s,\varepsilon}}\left(1 + \int R_s(k_{s,\varepsilon}|g|)\right) \le 1 + \varepsilon. \end{aligned}$$

Therefore, $|I(iy)| \le (1+\varepsilon)M_0 \ \ \forall y \in \mathbf{R}$. Similarly, $|I(1+iy)| \le (1+\varepsilon)M_1$. Hence by the three lines theorem, it follows that

$$|I| = |I(s)| \le (1+\varepsilon)M_0^{1-s}M_1^s, \qquad 0 \le s \le 1.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $|I| \le M_0^{1-s}M_1^s$. □

Remark. Rao (see [R, Theorems 1, 2, and 4]) originally proved an analog of Theorem 2 with the sharp constant $M_0^{1-s}M_1^s$ using the *Orlicz* norm on both sides in place of the *Luxemburg* norm. Since the two norms are equivalent ($\|f\|_Q \le \|f\|_Q^* \le 2\|f\|_Q$), it is easy to see that Rao's result implies (4) with the constant $M_0^{1-s}M_1^s$ replaced by $AM_0^{1-s}M_1^s$ with some constant $A > 1$ (see the remark following Theorem 2 in [R]; also see [M, p. 102] in this connection).

Corollary 4. *Let T be a multilinear operator such that for $p_{j,\ell} \in [1, \infty]$*

$$\|T(f_1, \dots, f_K)\|_Q \le M_\ell \prod_{j=1}^{K} \|f_j\|_{p_{j,\ell}} \qquad (\ell = 0, 1).$$

If $\frac{1}{p_j} = \frac{1}{p_{j,s}} = \frac{1-s}{p_{j,0}} + \frac{s}{p_{j,1}}$, $1 \le j \le K$, $0 \le s \le 1$, *then*

$$\|T(f_1, \dots, f_K)\|_Q \le M_0^{1-s} M_1^s \prod_{j=1}^{K} \|f_j\|_{p_j}.$$

Proof. Just take $Q_\ell = Q_s^+ = Q$, and $\Phi_{j,s}(t) = t^{p_{j,s}}$ (when $p_{j,s} = \infty$, take $\Phi_{j,s}(t) = 0$ for $t \le 1$, and $= \infty$ for $t > 1$) in Theorem 2. (In fact, in this case the arguments simplify considerably. So we may give a short direct proof with $g_z = g$ and

$$f_{j,z} = |f_j|^{p_j/p_j(z)} \cdot e^{i \arg(f_j)} \quad (\text{if } p_j < \infty),$$

where $\frac{1}{p_j(z)} = \frac{1-z}{p_{j,0}} + \frac{z}{p_{j,1}}$. (Let $f_{j,z} = f_j$, if $p_j = \infty$.) The rest proceeds exactly as in the proof of the Riesz-Thorin theorem (see [SW]) once we use Lemma 3 and Hölder's inequality (6).) □

Corollary 5. *Let* $K \ge 2$, $p \in [1, \infty)$, *where* $\frac{1}{p} = \sum_{j=1}^{K} \frac{1}{p_j}$, *and* $p_j \in (1, \infty]$. *Suppose* T *is a multilinear operator and that*

$$(\mathrm{E}_j) \qquad \|T(f_1, \dots, f_K)\|_Q \le M_j \|f_j\|_p \prod_{\ell \ne j} \|f_\ell\|_\infty \qquad (j = 1, \dots, K).$$

Then

$$\|T(f_1, \dots, f_K)\|_Q \le \prod_{j=1}^{K} (M_j^{p/p_j} \|f_j\|_{p_j}).$$

Proof. We use a straightforward induction argument. If $K = 2$, the corollary follows from Corollary 4. Now assume that the corollary is true with K replaced by $K - 1$ for some $K \ge 3$. Interpolating the estimates (E_K) and (E_j), using Corollary 4, gives for $1 \le j \le K - 1$

$$(\mathrm{F}_j) \qquad \|T(f_1, \dots, f_K)\|_Q \le M_K^{p/p_K} M_j^{p/q} \|f_K\|_{p_K} \|f_j\|_q \prod_{\ell \ne K, j} \|f_\ell\|_\infty,$$

where $\frac{1}{q} = \sum_{\ell=1}^{K-1} \frac{1}{p_\ell}$. We may assume $q < \infty$, since otherwise there is nothing to prove. Now fix a function f_K and apply the induction hypothesis to the estimates (F_j), $1 \le j \le K - 1$ (with the constants $\widetilde{M}_j = M_K^{p/p_K} M_j^{p/q} \|f_K\|_{p_K}$). We get

$$\begin{aligned} \|T(f_1, \dots, f_K)\|_Q &\le \prod_{j=1}^{K-1} (\widetilde{M}_j^{q/p_j} \|f_j\|_{p_j}) \\ &= M_K^{p/p_K} \|f_K\|_{p_K} \prod_{j=1}^{K-1} (M_j^{p/p_j} \|f_j\|_{p_j}) = \prod_{j=1}^{K} (M_j^{p/p_j} \|f_j\|_{p_j}). \quad \square \end{aligned}$$

3. Proof of Theorem 1

Fix a domain Ω in $\mathbf{R}^n$, $0 < |\Omega| < \infty$, and $p \in (1, \infty)$. Define a Young's function Q by $Q(t) = \frac{1}{c_1|\Omega|}\left(\exp(\frac{n}{w_{n-1}} t^{p'}) - 1\right)$, $t \ge 0$, where $c_1 = c_0 - 1$ (> 0)

and $c_0 = c_0(p)$ is the constant appearing in (1). Then (1) may be rewritten as

$$\int_\Omega Q\left(\frac{|I_\alpha(f)(x)|}{\|f\|_p}\right)dx \leq 1,$$

which is equivalent to the norm estimate

$$\|I_\alpha(f)\|_Q \leq \|f\|_p \quad (\forall f \in L^p(\Omega)), \tag{7}$$

in view of the definition of the (Luxemburg) norm. In exactly the same way, the estimate (3) is equivalent to

$$\|I_\alpha(f_1, \dots, f_K)\|_Q \leq L^{-1} \cdot \prod_{j=1}^{K} \|f_j\|_{p_j} \quad (\forall f_j \in L^{p_j}(\Omega),\ 1 \leq j \leq K). \tag{8}$$

We will deduce (8) from (7) and Corollary 5. We have for $1 \leq j \leq K$

$$\begin{aligned} |I_\alpha(f_1, \dots, f_K)(x)| &\leq \int_{\mathbf{R}^n} |f_j(x - \theta_j y)|\ |y|^{\alpha-n} dy \prod_{\ell \neq j} \|f_\ell\|_\infty \\ &= |\theta_j|^{-\alpha} I_\alpha(|f_j|)(x) \prod_{\ell \neq j} \|f_\ell\|_\infty. \end{aligned}$$

Hence

$$\begin{aligned} \|I_\alpha(f_1, \dots, f_K)\|_Q &\leq |\theta_j|^{-\alpha} \|I_\alpha(|f_j|)\|_Q \prod_{\ell \neq j} \|f_\ell\|_\infty \\ &\leq |\theta_j|^{-\alpha} \|f_j\|_p \prod_{\ell \neq j} \|f_\ell\|_\infty \quad \text{(by (7))} \quad (1 \leq j \leq K). \end{aligned}$$

Therefore, an application of Corollary 5 gives

$$\begin{aligned} \|I_\alpha(f_1, \dots, f_K)\|_Q &\leq \prod_{j=1}^{K} (|\theta_j|^{-\alpha p/p_j} \|f_j\|_{p_j}) \\ &= \prod (|\theta_j|^{-n/p_j} \|f_j\|_{p_j}) \quad (\text{since } \alpha = n/p) \\ &= L^{-1} \prod_{j=1}^{K} \|f_j\|_{p_j}, \end{aligned}$$

with $L = \prod_{j=1}^{K} |\theta_j|^{n/p_j}$. This finishes the proof of (8) and Theorem 1. □

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