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A DISCRETE FRACTAL IN $\mathbb{Z}^1_+$

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ABSTRACT. In this paper, we show that the level sets of mean zero finite variance random walks in $\mathbb{R}^1$ form a discrete fractal in the sense of Barlow and Taylor. Analogously to the Brownian motion result, the Hausdorff dimension of the level sets is almost surely equal to $\frac{1}{2}$.

1. INTRODUCTION

Let $Y_1, Y_2, \ldots$ be an independent sequence of identically distributed $\mathbb{Z}^d$-valued random variables with mean vector zero when it is defined. Assume that $d \geq 2$, and define the corresponding random walk $T_n = \sum_{j=1}^n Y_j$. Suppose there exist constants c_1 and c_2 such that, for all $x, y \in \mathbb{Z}^d$ with $x \neq y$,

$$c_1|x-y|^{\alpha-d} \leq \mathbb{E}^x \sum_n 1_{\{y\}}(T_n) \leq c_2|x-y|^{\alpha-d}$$

for some fixed $0 < \alpha \leq 2$ with $\alpha < d$. (In particular, the random walk is transient.) Then among other interesting results, in [BT2] Barlow and Taylor prove that the range of the walk $\{T_j; j \geq 1\}$ has a structure similar to that of a stable Lévy process with index α. More precisely, they proved that $\dim_H(\bigcup_{i=1}^\infty \{T_j\}) = \alpha$, almost surely. Here $\dim_H(A)$ denotes the discrete Hausdorff dimension of Borel set $A \subseteq \mathbb{Z}^d$ (see [BT1] or [BT2]). The above result shows that many of the dimension properties of stable processes in $\mathbb{R}^d$ have rigorous discrete analogues in $\mathbb{Z}^d$. Such ideas are useful and have appeared in the physics literature; see [N]. Motivated by this, one wants to know what happens in the case of recurrent random walks. We shall start by considering lattice walks. To this end, let $X_1, X_2, \ldots$ be i.i.d. $\mathbb{Z}^1$-valued random variables with mean zero and finite variance. Define the random walk $\xi_n = \sum_{j=1}^n X_j$. Unless specifically mentioned to the contrary, we always take ξ_0 to be zero. Let Z denote the "zero set" of ξ, i.e.,

$$Z = \{j \geq 1 : \xi_j = 0\}. \tag{1.1}$$

Following [BT2], define $\mathscr{H}$ to be the collection of all monotonically increasing functions, $h: \mathbb{R}^1_+ \mapsto \mathbb{R}^1_+$, satisfying $h(0) = 0$ and

$$h(2t) \leq K_h h(t) \quad \text{for all } 0 \leq t \leq \tfrac{1}{2}.$$

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Here K_h depends only on the function $h \in \mathscr{H}$ whose elements are said to be *measure functions*. Corresponding to every $h \in \mathscr{H}$ define, for all $A \subseteq \mathbb{Z}_+^1$,
(1.2)

$$m_h(A) = \sum_{n=1}^{\infty} \min \left\{ \sum_{j=1}^{m} h\left(\frac{\#(B_j \cap \mathbb{Z}_+^1)}{2^{n-2}}\right) : B_j \in \mathfrak{C},\ A \cap [2^{n-2}, 2^{n-1}) \subseteq \bigcup_{j=1}^{m} B_j \right\},$$

where $\mathfrak{C}$ is the collection of all intervals in $\mathbb{Z}_+^1$ which are of the form $[a, b)$, $a < b$ both in $\mathbb{Z}_+^1$. This is the discrete Hausdorff h-measure that appears in [BT1, BT2] specialized to the case of $\mathbb{Z}_+^1$. When $h(x) = x^\beta$ for $\beta > 0$, we write m_β for m_h. Based on this, one can define the (discrete) Hausdorff dimension of $A \subseteq \mathbb{Z}_+^1$ as

$$\dim_H(A) = \inf\{\beta > 0 \colon m_\beta(A) < \infty\}. \tag{1.3}$$

For properties of discrete Hausdorff measure and dimension, see [BT2].

The main results of this paper are the following:

Proposition 1.1. *Suppose $h \in \mathscr{H}$ satisfies*

$$\int_0^{e^{-e}} \frac{h(\varepsilon)\sqrt{\log\log(1/\varepsilon)}}{\varepsilon^{3/2}}\, d\varepsilon < \infty.$$

Then almost surely, $m_h(Z) < \infty$.

Proposition 1.2. *Let $h \in \mathscr{H}$ be given by $h(\varepsilon) = \sqrt{\varepsilon/(\log(1/\varepsilon) \vee 1)}$. Then almost surely, $m_h(Z) = \infty$.*

The following is immediate from Propositions 1.1 and 1.2.

Corollary 1.3. *With probability one,* $\dim_H(Z) = \frac{1}{2}$.

The above corollary shows that the discrete fractal index of Z is the same as the fractal dimension of the zero set of Brownian motion.

The proofs of Propositions 1.1 and 1.2 appear in §2. In §3 we first present an extension of Corollary 1.3 to some lattice-valued random walks in the domain of attraction of α-stable Lévy processes. Then we show how to prove a suitable restatement of Corollary 1.3 for all random walks in the real line which have mean zero and variance one. By scaling, this clearly implies the corresponding result for all mean zero finite variance random walks.

Finally it should be pointed out that we have used the term "(discrete) fractal" rather loosely. However, with little extra effort, our estimates imply that in the situation of Corollary 1.3, for example, $\dim_P(Z) = \frac{1}{2}$ almost surely. Here $\dim_P$ denotes the discrete packing dimension as defined in [BT2]. In other words, the above level sets are indeed fractals as defined by [BT2].

Throughout, $\# A$ will denote the cardinality of $A \subset \mathbb{Z}_+^1$, and $c_1, c_2, \ldots$ are constants whose value is unimportant and may change from line to line.

2. Proofs

Define for $0 < \lambda < 1$, $k \geq 1$, and $n \geq 0$,

$$\eta(n) = \sum_{j=0}^{n} 1_{\{0\}}(\xi_j) = \#(Z \cap [0, n]), \tag{2.1a}$$

$$u(\lambda) = \mathbb{E} \sum_{j=0}^{\infty} \lambda^j 1_{\{0\}}(\xi_j), \tag{2.1b}$$

$$\varrho(\lambda) = \mathbb{E}\lambda^{T_1}, \tag{2.1c}$$

$$T_k = \min\{n \geq 0 \colon \eta(n) = k\}. \tag{2.1d}$$

We start by recalling some basic identities and inequalities.

Lemma 2.1. *Suppose ξ is any random walk on $\mathbb{Z}^1$ (no restrictions on its mean nor variance). Then, for all $0 < \lambda < 1$ and all $k \geq 1$,*

$$\mathbb{E}\lambda^{T_k} = \left(\frac{u(\lambda)}{1 + u(\lambda)}\right)^k.$$

Proof. By the Markov property, $\{T_k ; k \geq 1\}$ is an increasing random walk; therefore, for all $\lambda \in (0, 1)$,

$$\mathbb{E}\lambda^{T_k} = \varrho(\lambda)^k. \tag{2.2}$$

Now we write

$$u(\lambda) = \mathbb{E} \sum_{j=0}^{\infty} \lambda^j (\eta(j) - \eta(j-1)).$$

Since $\eta(j) - \eta(j-1) = \sum_{k=1}^{\infty} 1_{\{j\}}(T_k)$, the above expression is equal to

$$u(\lambda) = \mathbb{E} \sum_{k=1}^{\infty} \lambda^{T_k} = \frac{\varrho(\lambda)}{1 - \varrho(\lambda)}.$$

The lemma follows from the above identity and (2.2). □

Remark 2.1.1. (1) If $\mathbb{E}X_1 = 0$ and $\mathbb{E}X_1^2 = 1$, using the saddle point method one can see that as $\lambda \to 1$

$$u(\lambda) \sim \sqrt{\frac{2}{1-\lambda}}.$$

(2) For any random walk ξ, the following holds:

$$\sup_{0<\lambda<1} (1 - \lambda^n)^{-1} (u(\lambda)^k (1 + u(\lambda))^{-k} - \lambda^n)$$
$$\leq \mathbb{P}(\eta(n) \geq k) \leq \inf_{0<\lambda<1} \left(\frac{u(\lambda)}{1 + u(\lambda)}\right)^k \lambda^{-n}.$$

For this and more, see [FP].

Corollary 2.2. *If* $\mathbb{E}X_1 = 0$ *and* $\mathbb{E}X_1^2 = 1$, *then almost surely*

$$\limsup_{n\to\infty} \frac{\eta(n)}{\sqrt{2n\log\log n}} = 1,$$

and

$$\limsup_{n\to\infty} n^{-1} \max_{j\le 2^{n-2}} \max_{3\le k\le 2^{n-2}} \sqrt{\frac{\log k}{k}}(\eta(k+j)-\eta(j)) < \infty.$$

Proof. The first statement is well known and is a consequence of Remarks 2.1.1(1), (2). Indeed (see [K, FP]), Remark 2.1.1 implies that, for all $\varepsilon > 0$, there exists $K(\varepsilon)$, $c(\varepsilon) > 0$, such that, for all $k \ge K(\varepsilon)$,

$$\mathbb{P}(\eta(n) \ge k) \le c(\varepsilon)\exp(-(1-\varepsilon)k^2/2n). \tag{2.3}$$

For the second result notice that by the strong Markov property and (2.3) for all $\varepsilon > 0$ and all $x > K(\varepsilon)$

$$\begin{aligned}
&\mathbb{P}\left(\max_{\substack{3\le k\le 2^{n-2}\\ j\le 2^{n-2}}} \sqrt{\log k/k}(\eta(j+k)-\eta(j)) \ge x\right)\\
&\quad\le 2^{2n-4}\max_{\substack{3\le k\le 2^{n-2}\\ j\le 2^{n-2}}} \mathbb{P}(\eta(j+k)-\eta(j) \ge x\sqrt{k/\log k})\\
&\quad\le 2^{2n}\max_{k\le 2^{n-2}} \mathbb{P}(\eta(k) \ge x\sqrt{k/\log k}) \le 2^{2n}c(\varepsilon)\exp(-\log 2(1-\varepsilon)x^2/2n).
\end{aligned}$$

The second result now follows from the Borel-Cantelli lemma. □

Proof of Proposition 1.1. Define $\tau_1 = \min\{j \ge 1 : \xi_j = 0\}$, $\tau_2 = \min\{j > \tau_1 : \xi_j = 0\}$, etc. Let $B_k = [\tau_k, \tau_k + 1)$. Then it is easy to see that $B_j \in \mathfrak{C}$ for all $j \ge 1$ and

$$Z \cap [2^{n-2}, 2^{n-1}) \subseteq \bigcup_{\{j\,:\, 2^{n-2}\le\tau_j<2^{n-1}\}} B_j. \tag{2.4}$$

Therefore, our choice of B_j given by (2.4) is a possible covering of $Z \cap [2^{n-2}, 2^{n-1})$. Substituting this in (1.2), we see from Corollary 2.2 that there exists an almost surely finite random variable V such that, with probability one,

$$\begin{aligned}
m_h(Z) &\le \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} h(2^{2-n})1_{[2^{n-2},2^{n-1})}(\tau_j)\\
&= \sum_{n=1}^{\infty} h(2^{2-n})(\eta(2^{n-1}-1) - \eta(2^{n-2}-1))\\
&\le c_3 V K_h \sum_{n=1}^{\infty} h(2^{-n})\sqrt{2^n \log\log 2^n} \quad (\text{since } h \in \mathscr{H})\\
&\le c_4 V \int_0^{e^{-e}} \frac{h(\varepsilon)\sqrt{\log\log(1/\varepsilon)}}{\varepsilon^{3/2}}\,d\varepsilon < \infty.
\end{aligned}$$

This finishes the proof of Proposition 1.1. □

Proof of Proposition 1.2. Let $B_{i,n} = [a_{i,n}, b_{i,n}) \cap \mathbb{Z}_+^1$ be any (possibly random) covering of $Z \cap [2^{n-2}, 2^{n-1})$. Corollary 2.2 implies that, for all $\varepsilon > 0$, there exists M such that, for all $n \geq M$,

$$h\left(\frac{\# B_{i,n}}{2^{n-2}}\right) \geq 2^{-(n-2)/2}\sqrt{\frac{\# B_{i,n}}{\log(2^{n-2}/\# B_{i,n})}}$$
$$\geq c_5(1-\varepsilon)n^{-1/2}2^{-n/2}(\eta(b_{i,n}) - \eta(a_{i,n})).$$

Therefore since $\bigcup_i B_{i,n} \supseteq [2^{n-2}, 2^{n-1}) \cap \mathbb{Z}_+^1$, for all $n \geq M$,

$$\sum_i h\left(\frac{\# B_{i,n}}{2^{n-2}}\right) \geq c_5(1-\varepsilon)n^{-1/2}2^{-n/2}(\eta(2^{n-1}-1) - \eta(2^{n-2})) \quad \text{a.s.}$$

Summing over n, we see that the proposition follows if we show that

$$F_N \equiv \sum_{j=1}^{N} \frac{\eta(2^j) - \eta(2^{j-1})}{\sqrt{j}2^{j/2}}$$

goes to infinity almost surely, as $N \to \infty$.

From Remark 2.1.1(1) and the Karamata Tauberian theorem [BGT, Corollary 1.7.3], for all $n \geq 1$,

$$c_6\sqrt{n} \leq \mathbb{E}\eta(n) \leq c_7\sqrt{n}. \tag{2.5}$$

Moreover, since $n \mapsto \eta(n)$ is a subadditive process whose jumps are of size one, it follows from (2.5) and [DM, Theorem VI.105.1] that

$$c_6^2 n \leq \mathbb{E}\eta(n)^2 \leq 2c_7^2 n. \tag{2.6}$$

Integrating by parts, we see that

$$F_N = (N2^N)^{-1/2}\eta(2^N) - 2^{-1/2}\eta(2) + \sum_{i=2}^{N-1} \eta(2^j)\left(\frac{1}{\sqrt{j2^j}} - \frac{1}{\sqrt{(j+1)2^{j+1}}}\right).$$

The above observation and (2.5) together imply

$$c_8\sqrt{N} \leq \mathbb{E}F_N \leq c_9\sqrt{N}. \tag{2.7}$$

Next, we proceed to estimate the second moment of F_N. Indeed by (2.7), (2.6), and the strong Markov property,

$$c_8^2 N \leq \mathbb{E}F_N^2 \leq c_{10}\log N + 2\sum_{i=1}^{N}\sum_{j=1}^{i-1} \frac{\mathbb{E}\eta(2^{j-1})\mathbb{E}\eta(2^{i-1})}{\sqrt{ij}2^{(i+j)/2}}$$
$$\leq c_{10}\log N + c_{11}\sum_{i=1}^{N}\sum_{j=1}^{i-1}(ij)^{-1/2} \leq c_{12}N.$$

Therefore,

$$c_8\sqrt{N} \leq \mathbb{E}F_N = \mathbb{E}F_N 1_{\{F_N \geq c_8\sqrt{N}/2\}} + \mathbb{E}F_N 1_{\{F_N < c_8\sqrt{N}/2\}}$$
$$\leq \sqrt{c_{12}N\mathbb{P}(F_N \geq c_8\sqrt{N}/2)} + c_8\sqrt{N}/2.$$

Therefore,

$$\mathbb{P}(F_N \geq c_8\sqrt{N}/2) \geq c_8^2/(4c_{12}) > 0.$$

Since $N \mapsto F_N$ is increasing, F_N must diverge in probability and by Kolmogorov's 0-1 law, almost surely. This concludes the proof. □

3. Extensions

(a) **Stable walks.** Suppose $X_1, X_2, \ldots$ are i.i.d. $\mathbb{Z}^1$-valued random variables and $\xi_n = \sum_{i=1}^n X_i$ is the corresponding random walk. Suppose further that, for some $1 < \alpha \leq 2$, the function $u(\lambda)$ as defined in (2.1b) satisfies the following for all $0 < \lambda < 1$:

$$c_{13}(1-\lambda)^{1-1/\alpha} \leq u(\lambda) \leq c_{14}(1-\lambda)^{1-1/\alpha}. \tag{3.1}$$

By adapting the proof of Karamata's theorem in [BGT, Corollary 1.7.3], one sees that (3.1) is equivalent to the following holding for all $k \geq 1$:

$$c_{15}k^{1-1/\alpha} \leq \mathbb{E}\eta(k) \leq c_{16}k^{1-1/\alpha}. \tag{3.2}$$

We then have the following analogue of the result for stable processes implied by [TW, Theorem 1]:

Proposition 3.1. *Suppose* (3.1) *holds and* Z *is defined by* (1.1). *Then almost surely,* $\dim_H(Z) = 1 - 1/\alpha$.

Remark 3.1.1. One can get more precise information on the exact measure function (as in Propositions 1.1 and 1.2) with extra effort. The needed estimates can be found, for example, in [MR].

Proof. By (3.2) and subadditivity for all $n, k \geq 1$,

$$\mathbb{E}\eta(k)^n \leq n!c_{16}^n k^{n(1-1/\alpha)}. \tag{3.3}$$

(See [DM, Theorem IV.105.1].) Therefore, for all $n \geq 1$,

$$\mathbb{E}\exp(n^{-(1-1/\alpha)}\eta(n)/2c_{16}) \leq 2. \tag{3.4}$$

By the Borel-Cantelli lemma,

$$\limsup_{n\to\infty} \frac{\eta(n)}{n^{1-1/\alpha}\log\log n} < \infty \quad \text{a.s.}$$

Arguing as in the proof of Proposition 1.1, we see that, if $h \in \mathscr{H}$ satisfies

$$\int_0^{e^{-e}} \frac{h(\varepsilon)\log\log(1/\varepsilon)}{\varepsilon^{2-1/\alpha}}\, d\varepsilon < \infty,$$

then $m_h(Z) < \infty$, almost surely. In particular, $\dim_H(Z) \leq 1 - 1/\alpha$, with probability one. To find the lower bound, use (3.4) and the strong Markov property as follows: for all $n \geq 3$, $3 \leq k \leq 2^{n-2}$, and all $\varepsilon, x > 0$,

$$\mathbb{P}(\eta(k+j) - \eta(j) \geq k^{1-1/\alpha-\varepsilon}x) \leq c_{17}e^{-x}.$$

Therefore, as in the proof of Proposition 1.2 with probability one,

$$\max_{3\leq k\leq 2^{n-2}} \max_{j\leq 2^{n-2}} k^{-(1-1/\alpha-\varepsilon)}(\eta(k+j) - \eta(j)) \leq c_{18}n,$$

eventually. Continuing through the proof of Proposition 1.2, picking $h \in \mathscr{H}$ by $h(x) = x^{1-1/\alpha-\varepsilon}$, it follows that $m_h(Z) = \infty$, almost surely, for all $\varepsilon > 0$. The lower bound on the dimension is the consequence of (1.3). □

(b) **Nonlattice case.** Suppose $X_1, X_2, \ldots$ are i.i.d. random variables with mean zero and variance one. Let $\varphi(t) = \mathbb{E}\exp(itX_1)$ be their characteristic function. If the X's are nonlattice, i.e., $|\varphi(t)| = 1$ is equivalent to $t = 0$, then the random walk $\xi_n = \sum_{i=1}^n X_i$ does not hit zero. In this case, the correct notion of the zero set is given by the set of *close approaches*,

$$Z = \{j \geq 1 : |\xi_j| \leq \tfrac{1}{2}\}. \tag{3.5}$$

Of course, in the lattice case, this matches with (1.1).

Proposition 3.2. *Under the above assumptions,* $\dim_H(Z) = \frac{1}{2}$, *almost surely.*

Remark 3.2.1. It is well known that distributions on the real line are either lattice or nonlattice. Therefore, the above, together with Corollary 1.3, imply that the discrete Hausdorff dimension of the level sets of any mean zero variance one random walk on the real line is one half.

Proof. Define the local time of ξ by $\eta(n) = \sum_{i=0}^n 1_{(-1/2,1/2)}(\xi_j)$. The proof is almost identical to that of Proposition 3.1. The only estimates that one needs to prove, at this point, are upper and lower estimates for $\sup_{|x|<1/2} \mathbb{E}(\eta(k)|\xi_0 = x)$. This is needed to make all the strong Markov arguments work (in particular, the argument that leads from (3.2) to (3.3)). Fortunately, such estimates are well known. Indeed by [S, Theorem 1], for all $k \geq 1$,

$$\sup_x \left| \mathbb{P}(|\xi_k| \leq \tfrac{1}{2}|\xi_0 = x) - (2k\pi)^{-1/2} \int_{-1/2}^{1/2} e^{-(x-u)^2/2k}\, du \right| = o(k^{-1/2}).$$

Summing the above from $k = 1$ to n, we have

$$c_{16}\sqrt{n} \leq \sup_{|x|<1/2} \mathbb{E}(\eta(n)|\xi_0 = x) \leq c_{20}\sqrt{n}.$$

The rest of the modifications are standard. □

References

[BGT] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation* (J. C. Rota, ed.), vol. 27, Cambridge Univ. Press, 1987.

[BT1] M. T. Barlow and S. J. Taylor, *Fractal dimension of sets in discrete spaces*, J. Phys. A **22** (1989), 2621–2626.

[BT2] ——, *Defining fractal subsets of* $\mathbb{Z}^d$, Proc. London Math. Soc. (3) **64** (1992), 125–152.

[DM] C. Dellacherie and P. A. Meyer, *Probabilities and potential B*, North-Holland Math. Stud., vol. 72, North Holland, Amsterdam, 1982.

[FP] B. Fristedt and W. Pruitt, *Lower functions for increasing random walks and subordinators*, Z. Wahrsch. Verw. Gebiete **18** (1971), 167–182.

[K] H. Kesten, *An iterated logarithm law for local time*, Duke Math. J. **32** (1965), 447–456.

[MR] M. B. Marcus and J. Rosen, *Laws of the iterated logarithm for the local times of symmetric Lévy processes and recurrent random walks*, preprint.

[N] J. Naudts, *Dimension of discrete fractal spaces*, J. Phys. A **21** (1988), 447–452.

[S] C. J. Stone, *A local limit theorem for nonlattice multi-dimensional distribution functions*, Ann. Math. Statist. **36** (1965), 546–551.

[TW] S. J. Taylor and J. G. Wendel, *The exact Hausdorff measure of the zero set of a stable process*, Z. Wahrsch. Verw. Gebiete **6** (1966), 170–180.

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