PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 2, February 1994

A NOTE ON SUBCONTINUA OF $\beta[0, \infty) - [0, \infty)$

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(Communicated by Franklin D. Tall)

ABSTRACT. Let $M = \bigoplus_{n\in\omega} I_n$ be the topological sum of countably many copies of the unit interval I. For any ultrafilter $u \in \omega^*$, we let $M^u = \bigcap\{\mathrm{cl}_{\beta M}(\bigcup\{I_n : n \in A\}) : A \in u\}$. It is well known that M^u is a decomposable continuum with a very nice internal structure. In this paper, we show:

(1) every nondegenerate subcontinuum of $\beta[0, \infty) - [0, \infty)$ contains a copy of M^u for some $u \in \omega^*$.

(2) there is no nontrivial simple point in Laver's model for the Borel conjecture.

The second answers a question posed by Baldwin and Smith negatively.

0. INTRODUCTION

In this paper we study subcontinua of the Stone-Čech compactification of the nonnegative reals. We refer to [9] and [12] for background on this topic. The unit interval $[0, 1]$ is denoted by I. Let $I_n = I \times \{n\}$ for $n \in \omega$, and let $M = \bigoplus_{n\in\omega} I_n$ be the topological sum. For any ultrafilter $u \in \omega^*$, we let

$$M^u = \bigcap\{\mathrm{cl}_{\beta M}(\bigcup\{I_n : n \in A\}) : A \in u\}.$$

It is not difficult to prove that M^u is a continuum (see, e.g., [5]). If we let $i: M \to \omega$ be the map defined by $i(r) = n$ for any $r \in I_n$ and $\beta i: \beta M \to \beta\omega$ be the extension of i, it is easy to see that $M^u = \beta i^{-1}(u)$. So every subcontinuum of $\beta M - M$, and therefore, every proper subcontinuum of $\beta[0, \infty) - [0, \infty)$, can be embedded into M^u for some $u \in \omega^*$. Moreover, we have

Theorem 1. *Every nondegenerate subcontinuum of $\beta[0, \infty) - [0, \infty)$ contains a copy of M^u for some $u \in \omega^*$.*

For any map $f \in^\omega I$ and $u \in \omega^*$, let $f^u = \{F \subset M : F$ is closed and $\{n : (f(n), n) \in F\} \in u\}$ and $P^u = \{f^u : f \in^\omega I\}$. It is well known that f^u is a cut point of M^u if $\{n \in \omega : f(n) \neq 0, 1\} \in u$ [9, (1)]. It is also well known that there are many indecomposable subcontinua with cardinalities 2^c in M^u for any $u \in \omega^*$ [9, (19)]. Therefore, by Theorem 1, we have

Received by the editors June 20, 1991 and, in revised form, May 11, 1992.
1991 *Mathematics Subject Classification.* Primary 54D40, 54F15, 03E35.
Key words and phrases. Stone-Čech remainder, Laver real, continuum.

Corollary. (a) *Every subcontinuum of $\beta[0, \infty) - [0, \infty)$ contains an indecomposable subcontinuum.*

(b) *$\beta[0, \infty)$ does not contain a nondegenerate hereditarily indecomposable subcontinuum.*

Assertion (a) is due to Bellamy [2]. (b) was proved by Smith in [10] (van Douwen also announced it in [4]). The following problem was first posed by van Douwen (see the remarks at the end of [11]).

Question 1 (van Douwen). Is there any cut point of M^u which is not in P^u?

Definition 1. A point $x \in \beta M$ is said to be (nontrivial) simple if for any $F \in x$ there is $U \in x$ such that $U \subset F$ and $U \cap I_n = \varnothing$ or $U \cap I_n$ is a (nondegenerate) interval.

Fact 1. (a) [12, §1, Corollary] *If x is a cut point of M^u and $x \notin P^u$, then x is a far point of βM.*

(b) [12, Theorem 1.1] *$x \in M^u$ is a nontrivial simple point if and only if x is a cut point of M^u and remote point of βM.*

The author [12] proved under CH that there is $u \in \omega^*$ such that there is a cut point of M^u which is not simple. Baldwin and Smith [1] proved that $\mathrm{MA}_{\text{countable}}$ implies that there is a nontrivial simple point. They asked

Question 2 (Baldwin and Smith [1]). Is there any nontrivial simple point in ZFC?

Theorem 2. *There is no nontrivial simple point in Laver's model for the Borel conjecture.*

We would like to mention that the conclusion in Theorem 2 holds in any extension of the model of ZFC + CH by ω_2-iteration with countable support of nontrivial proper forcing notions which add dominating reals and have the Laver property (see [6]).

Question 1 remains open!

1. Proof of Theorem 1

Let $X = [0, \infty)$ and $K \subset \beta X - X$ be a nondegenerate subcontinuum. The following lemma was proved by Smith in [10] for locally compact, locally connected metric spaces. We give a direct proof here.

Lemma 1.1. *Let $\{U_0, U_1, \ldots, U_m\}$ be a finite open cover of K in βX such that $U_i \cap K \neq \varnothing$ for all $i \leq m$. Then there is a closed interval $H \subset X$ such that $H \cap U_i \neq \varnothing$ for $i \leq m$ and $H \subset \bigcup\{U_i : i \leq m\}$.*

Proof. Let $V = \bigcup\{U_0, U_1, \ldots, U_m\}$ and $V' = V \cap X$. Then there are disjoint open intervals $\{J_n : n \in \omega\}$ so that $V' = \bigcup\{J_n : n \in \omega\}$. Let $A_0 = \{n \in \omega : J_n \cap U_0 \neq \varnothing\}$, $V_0 = \bigcup\{J_n : n \in A_0\}$, and $W_0 = \bigcup\{J_n : n \notin A_0\}$. We have $K \subset V \subset (\mathrm{cl}_{\beta X} V_0) \cup (\mathrm{cl}_{\beta X} W_0)$ and $(\mathrm{cl}_{\beta X} V_0) \cap (\mathrm{cl}_{\beta X} W_0) \subset (\mathrm{cl}_{\beta X} \overline{V}_0) \cap (\mathrm{cl}_{\beta X} \overline{W}_0) = \mathrm{cl}_{\beta X}(\overline{V}_0 \cap \overline{W}_0)$, where $\overline{V}_0$ and $\overline{W}_0$ are the closures of V_0 and W_0 in X respectively. Since V is an open neighbourhood of K, we have $K \cap (\mathrm{cl}_{\beta X}(\overline{V}_0 \cap \overline{W}_0)) = \varnothing$; therefore, $K \subset \mathrm{cl}_{\beta X} V_0$ since K is connected and $K \cap (\mathrm{cl}_{\beta X} V_0) \supset K \cap U_0 \neq \varnothing$.

If we let $A_i = \{n \in \omega : J_n \cap U_j \neq \varnothing \text{ for } j \leq i\}$ and $V_i = \bigcup\{J_n : n \in A_i\}$ for $i \leq m$, we can easily show by induction that $K \subset \mathrm{cl}_{\beta X} V_i$ for $i \leq m$; so $A_m \neq \varnothing$. This completes the proof of Lemma 1.1.

We let U_0 and U_1 be disjoint open sets of βX so that $(\mathrm{cl}_{\beta X} U_0) \cap (\mathrm{cl}_{\beta X} U_1) = \varnothing$ and $U_i \cap K \neq \varnothing$ $(i = 0, 1)$. Let $\mathscr{B}$ be the collection of closed intervals so that an interval $[a, b]$ belongs to $\mathscr{B}$ if and only if the following conditions hold:

(1) $[a, b] \cap (U_0 \cup U_1) = \varnothing$ and $a \neq b$,
(2) $\{a, b\} \subset \mathrm{Br}(U_0 \cap X) \cup \mathrm{Br}(U_1 \cap X)$ and $a \in \mathrm{Br}(U_0 \cap X)$ if and only if $b \in \mathrm{Br}(U_1 \cap X)$,

where Br denotes the boundary operation in X. Since $\mathrm{cl}_{\beta X} U_0$ and $\mathrm{cl}_{\beta X} U_1$ are disjoint, $\mathscr{B}$ is discrete. We enumerate $\mathscr{B}$ as $\{J_n : n \in \omega\}$. We need only show that there is $u \in \omega^*$ such that $\bigcap\{\mathrm{cl}_{\beta X}(\bigcup\{J_n : n \in A\}) : A \in u\} \subset K$. Let $\mathscr{U}$ be an open neighbourhood base of K in βX. For $U \in \mathscr{U}$, we let $A_U = \{n \in \omega : J_n \subset U\}$. Then $A_U \neq \varnothing$ for $U \in \mathscr{U}$, since the H given by Lemma 1.1 for the cover $\{U, U_0, U_1\}$ clearly contains a J_n for some $n \in A_U$. Since $A_U \subset A_V$ for $U \subset V$ and $U, V \in \mathscr{U}$, $\{A_U : U \in \mathscr{U}\}$ has finite intersection property. Let

$$M_{\mathscr{U}} = \bigcap\{\mathrm{cl}_{\beta X}(\bigcup\{J_n : n \in A_U\}) : U \in \mathscr{U}\}.$$

It is immediate that $M_{\mathscr{U}} \subset \bigcap\{\mathrm{cl}_{\beta M} U : U \in \mathscr{U}\} = K$. Note that

$$\mathrm{cl}_{\beta X}(\bigcup\{J_i : i \leq n\}) \cap K = \varnothing \quad \text{for } n \in \omega.$$

So if u is an ultrafilter on ω and $\{A_U : U \in \mathscr{U}\} \subset u$, then $u \in \omega^*$ and

$$\bigcap\{\mathrm{cl}_{\beta X}(\bigcup\{J_n : n \in A\}) : A \in u\} \subset K.$$

This completes the proof of Theorem 1.

2. Proof of Theorem 2

Recall that there is a natural partial order $<_u$ on M^u for $u \in \omega^*$ defined as follows: $x <_u y$ if and only if there are $F \in x$ and $H \in y$ such that $\{n \in \omega : F \cap I_n < H \cap I_n\} \in u$, where $F \cap I_n < H \cap I_n$ means that $r < s$ for all $(r, n) \in F \cap I_n$ and $(s, n) \in H \cap I_n$. It is easily seen that $(P^u, <_u)$ is a linearly ordered set. In fact, $(P^u, <_u)$ is isomorphic to the ultrapower $({}^{\omega}I/u, <_u)$. Moreover, if $x, y \in M^u$ and $x <_u y$, then there is a $p \in P^u$ such that $x <_u p <_u y$. For $f, g \in^{\omega} I$ and $A \in u$, we let

$$[f, g; A] = \bigcup\{[f(n), g(n)] \times \{n\} : n \in A\}$$

and

$$[f, g; u] = \bigcap\{\mathrm{cl}_{\beta M}([f, g; A]) : A \in u\}.$$

It is easily seen that $M^u = [\mathbb{0}, \mathbb{1}; u]$, where $\mathbb{0}(n) = 0$ and $\mathbb{1}(n) = 1$ for $n \in \omega$. $[f, g; u]$ is a continuum since it is homeomorphic to M^u if $f^u <_u g^u$. For $x \in M^u$, we let

$$[x]_u = \{y \in M^u : y \text{ and } x \text{ are } <_u\text{-incomparable, or } x = y\}.$$

Note that $[f, g; u] = \{y \in M^u : f^u \leq_u y \leq_u g^u\}$ for $f, g \in^\omega I$, therefore, we have that for $x \in M^u$

$$(1) \qquad [x]_u = \bigcap\{[f, g; u] : f^u \leq_u x \leq_u g^u\}.$$

We can also prove that

$$(2) \qquad \begin{aligned} [x]_u = &\left(\mathrm{cl}_{\beta M}\left(\bigcup\{[f, g; u] : f^u \leq_u g^u <_u x\}\right)\right) \\ &\bigcap \left(\mathrm{cl}_{\beta M}\left(\bigcup\{[f, g; u]; x <_u f^u \leq g^u\}\right)\right) \end{aligned}$$

(see [9, (16); 12, Lemma 1.2]).

Lemma 2.1. (a) *A point* $x \in M^u$ *is a nontrivial simple point if and only if* $\{[f, g; A] : f, g \in^\omega I,\ A \in u, \text{ and } f^u <_u x <_u g^u\}$ *is a filter base for* x.

(b) *A point* $x \in M^u$ *is a cut point if and only if* $\{[f, g; A] : f, g \in^\omega I,\ A \in u, \text{ and } f^u <_u x <_u g^u\}$ *is a neighbourhood base for* x.

Proof. (a) is obvious. For (b), we assume that $\mathbb{0}^u <_u x <_u \mathbb{1}^u$. It follows easily from (1) that $[x]_u = \{x\}$ if and only if $\{\mathrm{cl}_{\beta M}([f, g; A]) : f, g \in^\omega I,\ A \in u, \text{ and } f^u <_u x <_u g^u\}$ is a neighbourhood base of x. Note that in (2) $\mathrm{cl}_{\beta M}(\bigcup\{[f, g; u] : f, g \in^\omega I \text{ and } f^u \leq_u g^u <_u x\})$ and $\mathrm{cl}_{\beta M}(\bigcup\{[f, g; u]; f, g \in^\omega I \text{ and } x <_u f^u \leq_u g^u\})$ are connected. Therefore, if $[x]_u = \{x\}$, x is a cut point of M^u; if $[x]_u \neq \{x\}$, then x is not a cut point of M^u, since $M^u \backslash \{x\}$ is connected.

For any $C \subset^\omega I$ and $u \in \omega^*$, we let $C^u = \{f^u \in P^u : f \in C\}$. We say that a pair $\mathscr{C} = (C, D)$ of subsets of ${}^\omega I$ determines a cut point x in M^u for some $u \in \omega^*$ if the following two conditions hold:

(1) $C^u <_u D^u$, i.e., $f^u <_u g^u$ for all $f \in C$ and $g \in D$.
(2) $\{\mathrm{cl}_{\beta M}([f, g; A]) : f \in C,\ g \in D,\ A \in u, \text{ and } f^u <_u x <_u g^u\}$ is a neighbourhood base for x.

It is easily seen that for a cut point x of M^u, if $C' \subset C$ and $D' \subset D$ satisfy that C'^u is cofinal in C^u and D'^u is coinitial in D^u, then (C, D) determines x if and only if (C', D') determines x. Note that Lemma 2.1(b) says that every cut point of M^u is determined by a pair of subsets of ${}^\omega I$.

Lemma 2.2. *Let* $\mathfrak{M} \subset \mathfrak{N}$ *be transitive models of ZFC such that there is* $r \in {}^\omega\omega \cap \mathfrak{N}$ *dominating every* $h \in {}^\omega\omega \cap \mathfrak{M}$, *i.e.,* $h(n) < r(n)$ *for all but finitely many* $n \in \omega$. *If* u *is a nonprinciple ultrafilter on* ω *in* $\mathfrak{N}$, *then no cut point of* M^u *is determined by a pair of subsets of* ${}^\omega I$ *in* $\mathfrak{M}$.

Proof. Let $\mathscr{P} = \bigcup_{n\in\omega}\mathscr{P}_n$ be a collection of closed rational subintervals of the unit interval I such that $\mathscr{P}_n$ is finite, pairwise disjoint and, for any interval $J \subset I$, if the length of J is larger than $1/n$, then $|\{H \in \mathscr{P}_n : H \subset J\}| \geq 2$. For $n \in \omega$, we enumerate $\mathscr{P}_{r(n)}$ as $\{H_{n,j} : j < m_n\}$ so that $H_{n,j-1} < H_{n,j}$, i.e., the right end of $H_{n,j-1}$ is less than the left end of $H_{n,j}$, for $j < m_n$. Let

$$F_i = \bigcup\{H_{n,2j+i} : n \in \omega \text{ and } 2j + i < m_n\}$$

for $i = 0, 1$. Then $F_0 \cap F_1 = \varnothing$. Suppose that $\mathscr{C} = (C, D)$ is a pair of subsets of ${}^\omega I$ in $\mathfrak{M}$. For $(f, g) \in C \times D$, we define $h_{f,g}\colon \omega \to \omega$ by

$$h_{f,g}(n) = \begin{cases} \min\{m : |f(n) - g(n)| > 1/m\} & \text{if } f(n) \neq g(n), \\ 0 & \text{otherwise}. \end{cases}$$

Then $h_{f,g} \in \mathfrak{M}$. If $C^u < D^u$, then for $A \in u$ and $i = 0, 1$ we take $[f, g; A] \cap F_i \neq \varnothing$, since $h_{f,g}(n) < r(n)$ for all but finitely many $n \in \omega$. Therefore, no cut point of M^u is determined by $\mathscr{C}$.

Let $\mathbb{P}_{\omega_2}$ be the ω_2-iteration of the Laver forcing with countable support, and let $\mathbb{G}_{\omega_2}$ be $\mathbb{P}_{\omega_2}$-generic over V. We assume that the continuum hypothesis holds in V. It is well known that a Laver real dominates every real in the ground model; therefore, by [3, Lemma 5.10; 7, Lemma 11], we have

Corollary 2.3. *There is no cut point of M^u determined by a pair of subsets of ${}^{\omega}I$ with cardinalities ω_1 in $V[\mathbb{G}_{\omega_2}]$ for any $u \in \omega^*$.*

The following property of the Laver forcing is weaker than the Laver property in [6]. It is well known that both the Laver forcing and the Mathias forcing have the Laver property and the Laver property is preserved under countable support iterated forcing. We refer to [6] for details.

Lemma 2.4 [6]. *For every $\mathbb{P}_{\omega_2}$-name $\dot{f}$ for a function from ω to ω and for every $p \in \mathbb{P}_{\omega_2}$, if $p \Vdash$ "$\dot{f}(n) < 2^{n^3}$" then there exist an extension q of p and a sequence $\{F_n : n \in \omega\}$ of finite subsets of ω in V such that $q \Vdash$ "$\dot{f}(n) \in F_n$" and $|F_n| < 2^{n^2}$ for $n \in \omega$.*

Lemma 2.5. *Suppose that $p \Vdash_{\mathbb{P}_{\omega_2}}$ "$\dot{f}: \omega \to I$". There are an extension q of p and a sequence $\{c_n : n \in \omega\}$ of codes for closed nowhere dense sets in V such that $q \Vdash |_{\mathbb{P}_{\omega_2}}$"$\dot{f}(n)$ belongs to the set coded by c_n for $n \in \omega$".*

Proof. We modify Miller's argument in [7, §6]. As usual, we deal with the Cantor space 2^{ω}, with the product topology, and the product measure μ on it. We work in $V[\mathbb{G}_{\omega_2}]$. Assume that $f: \omega \to 2^{\omega}$ is a sequence of reals. For $n \in \omega$, we define $\pi_n: (2^{\omega})^{\omega} \to 2^{\omega}$ by $\pi_n((x_n)) = x_n$. Let $X = \{x \in {}^{\omega}\omega : \forall n \in \omega(x(n) < 2^{n^3})\}$. We fix the bijection $e: \omega \times \omega \to \omega$ such that $e(n, m) < e(n, m+1)$ for $n, m \in \omega$. Define $\theta: X \to (2^{\omega})^{\omega}$ by

$$\theta(x)(n) = x(e(n, 0))^\frown x(e(n, 1))^\frown \cdots,$$

where we identify 2^{n^3} with sequences of 0's and 1's of length n^3. By Lemma 2.4 there is a sequence $\{F_n : n \in \omega\}$ of finite subsets of ω in V such that $\theta^{-1}(f)(n) \in F_n$ for $n \in \omega$. Let $C = \{x \in X : \forall n \in \omega(x(n) \in F_n)\}$. Then $f(n) \in \pi_n(\theta(C))$ for $n \in \omega$. It is easily seen that $\pi_n(\theta(C))$ is a closed subset of 2^{ω} and

$$\mu(\pi_n(\theta(C))) = \lim_{m \to \infty} \frac{|F_{e(n,m)}|}{2^{e(n,m)^3}} = \lim_{m \to \infty} \frac{1}{2^{e(n,m)}} = 0.$$

This completes the proof since every closed measure zero set is nowhere dense.

Corollary 2.5. *Let $x \in M^u$ be a nontrivial simple point and $\mathscr{C} = (C, D)$ a pair of subsets of ${}^{\omega}I$ determining x. Then in $V[\mathbb{G}_{\omega_2}]$, for any $u' \in \omega^*$ and $u \subset u'$, there is no $h \in {}^{\omega}I$ such that $C^{u'} < \{h^{u'}\} < B^{u'}$.*

Proof. Let $h \in {}^{\omega}I \cap V[\mathbb{G}_{\omega_2}]$. Then by Lemma 2.4 there is a sequence $\{c_n : n \in \omega\} \in V$ of codes for closed nowhere dense sets of I such that $h(n) \in \mathrm{eval}(c_n)$ for $n \in \omega$. Working in V, let $F = \bigcup\{\mathrm{eval}^V(c_n) \times \{n\} : n \in \omega\}$. Then F is nowhere dense in M. Therefore, there are $f \in C$, $g \in D$, and $A \in u$ such that $[f, g; A] \cap F = \varnothing$, since x is a nontrivial simple point and is determined by $\mathscr{C}$. So in $V[\mathbb{G}_{\omega_2}]$, $h^{u'} <_{u'} f^{u'}$ or $g^{u'} <_{u'} h^{u'}$ for any $u' \in \omega^*$ and $u \subset u'$.

Now we are in a position to complete the proof of Theorem 2. Suppose there there is a nontrivial simple point $x \in M^u$ in $V[\mathbb{G}_{\omega_2}]$. Then there is a pair $\mathscr{C} = (C, D)$ of subsets of ${}^\omega I$ determining x. By Lemma 5.10 in [3] there is $\alpha < \omega_2$ such that, in $V[\mathbb{G}_\alpha]$, x' is a nontrivial simple point of $M^{u'}$ and $\mathscr{C}' = (C', D')$ determines x', where $x' = x \cap V[\mathbb{G}_\alpha]$, $u' = u \cap V[\mathbb{G}_\alpha]$, $C' = C \cap V[\mathbb{G}_\alpha]$, and $D' = D \cap V[\mathbb{G}_\alpha]$. By [7, Lemma 11, Corollary 2.5], C'^u is cofinal in C^u and D'^u is coinitial in D^u. Therefore, $\mathscr{C}'$ determines x in $V[\mathbb{G}_{\omega_2}]$. This is impossible by Corollary 2.3.

Note added in proof

We refer to the author's *Continua in* R^*, Topology Appl. **50** (1993), 183–197, for more information. In particular, a consistent answer to Question 1 has been given by A. Dow and K. P. Hart.

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