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FINITE INDEX SUBFACTORS AND HOPF ALGEBRA CROSSED PRODUCTS

WOJCIECH SZYMAŃSKI

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ABSTRACT. We show that if $\mathbf{N} \subseteq \mathbf{M} \subseteq \mathbf{L} \subseteq \mathbf{K}$ is a Jones's tower of type II_1 factors satisfying $[\mathbf{M}: \mathbf{N}] < \infty$, $\mathbf{N}' \cap \mathbf{M} = \mathbb{C}I$, $\mathbf{N}' \cap \mathbf{K}$ a factor, then $\mathbf{M}' \cap \mathbf{K}$ bears a natural Hopf $*$-algebra structure and there is an action of $\mathbf{M}' \cap \mathbf{K}$ on $\mathbf{L}$ such that the resulting crossed product is isomorphic to $\mathbf{K}$.

1. INTRODUCTION

Since Jones's fundamental work [4], understanding the structure of subfactors of type II_1 factors has become one of the most important subjects in von Neumann algebra theory. Finite index subfactors with trivial relative commutant are of particular interest.

Several authors have tried to explain the relationship between subfactors and the crossed product construction related to group or, more generally, Hopf algebra actions. Treating the hyperfinite factor case Ocneanu has provided in [5] a very interesting and general scheme for studying this problem.

In this paper we give a simple proof of the following theorem, which was also announced by Ocneanu. If $\mathbf{N} \subseteq \mathbf{M} \subseteq \mathbf{L} \subseteq \mathbf{K}$ is a Jones's tower of type II_1 factors with finite index, $\mathbf{N}' \cap \mathbf{M} = \mathbb{C}$, and $\mathbf{N}' \cap \mathbf{K}$ a factor, then $\mathbf{M}' \cap \mathbf{K}$ has a natural Hopf $*$-algebra structure and acts on $\mathbf{L}$ in such a way that the resulting crossed product is isomorphic to $\mathbf{K}$. This can serve as an intrinsic characterization of crossed products of type II_1 factors by outer actions of finite-dimensional Hopf $*$-algebras (the downward construction plays a role).

A similar theorem for infinite index inclusion is conjectured in [3].

The preliminary section of our article contains some elementary and probably well-known facts. Next we give a detailed construction of a Hopf algebra structure on $\mathbf{M}' \cap \mathbf{K}$. The comultiplication is defined as a map dual to the multiplication in $\mathbf{N}' \cap \mathbf{L}$, duality between the two relative commutants being established by a very natural bilinear form.

In our approach we managed to avoid all cohomological complications.

As a byproduct we obtained a sharper version of the Pimsner-Popa trace inequality [6, Proposition 1.9].

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In the last section we give a simple formula for the action and prove the isomorphism theorem.

2. Preliminaries

Let $I \in \mathbf{N} \subseteq \mathbf{M}$ be type II_1 factors with finite index $[\mathbf{M}:\mathbf{N}] = \lambda$ and trivial relative commutant $\mathbf{N}' \cap \mathbf{M} = \mathbb{C}I$. Let $\mathbf{L}$ be the Jones extension of $\mathbf{M}$ by $\mathbf{N}$, that is, $\mathbf{L} = \langle \mathbf{M}, e_{\mathbf{N}} \rangle$, where $e_{\mathbf{N}}$ is the Jones projection. Similarly, let $\mathbf{K}$ be the Jones extension of $\mathbf{L}$ by $\mathbf{M}$ with the corresponding projection $e_{\mathbf{M}}$. By Proposition 3.1.7 of [4] we have $[\mathbf{K}:\mathbf{L}] = [\mathbf{L}:\mathbf{M}] = \lambda$, and it is clear that $\mathbf{M}' \cap \mathbf{L} = \mathbf{L}' \cap \mathbf{K} = \mathbb{C}I$.

We denote by τ the canonical trace on $\mathbf{K}$ and by $\mathbf{E_L}$ and $\mathbf{E_M}$ the conditional expectations related to τ from $\mathbf{K}$ onto $\mathbf{L}$ and $\mathbf{M}$ respectively. Clearly $\mathbf{E_M} = \mathbf{E_M} \circ \mathbf{E_L}$.

We denote $\mathbf{N}' \cap \mathbf{L}$ by $\mathbf{A}$, $\mathbf{M}' \cap \mathbf{K}$ by $\mathbf{B}$, and $\mathbf{N}' \cap \mathbf{K}$ by $\mathbf{C}$. All of them are finite dimensional $\mathbf{C}^*$-algebras (cf. [4, Corollary 2.2.3]). Proposition 1.9 of [6] implies that $e_{\mathbf{N}}$ and $e_{\mathbf{M}}$ are minimal and central projections in $\mathbf{A}$ and $\mathbf{B}$ respectively.

We denote by $\mathbf{D}$ the two-sided ideal of $\mathbf{C}$ generated by $e_{\mathbf{M}}$ (which coincides with the ideal generated by $e_{\mathbf{N}}$). We denote by $\mathbf{E_A}$ and $\mathbf{E_B}$ the conditional expectations related to τ from $\mathbf{C}$ onto $\mathbf{A}$ and $\mathbf{B}$ respectively. For any $c \in \mathbf{C}$ we denote by $\overline{c}$ its image under the projection of $\mathbf{C}$ onto $\mathbf{D}$.

The following proposition is easily established by considering the tower of commutants $\mathbf{K}' \subseteq \mathbf{L}' \subseteq \mathbf{M}' \subseteq \mathbf{N}'$, with $\mathbf{L}$ represented on $L^2(\mathbf{L}, \tau)$.

Proposition 1. *There is a tower of type* II_1 *factors* $I \in \mathbf{P} \subseteq \mathbf{Q} \subseteq \mathbf{R} \subseteq \mathbf{S}$ *such that*:

(1) $[\mathbf{Q}:\mathbf{P}] = \lambda$;
(2) $\mathbf{P}' \cap \mathbf{Q} = \mathbb{C}I$;
(3) $\mathbf{R}$ *is the Jones extension of* $\mathbf{Q}$ *by* $\mathbf{P}$ (*with the corresponding projection* $e_{\mathbf{P}}$);
(4) $\mathbf{S}$ *is the Jones extension of* $\mathbf{R}$ *by* $\mathbf{Q}$ (*with the corresponding projection* $e_{\mathbf{Q}}$); *and*
(5) *there is an isomorphism* $\theta: \mathbf{P}' \cap \mathbf{S} \to \mathbf{C}$ *such that* $\theta(\mathbf{P}' \cap \mathbf{R}) = \mathbf{B}$, $\theta(\mathbf{Q}' \cap \mathbf{S}) = \mathbf{A}$, $\theta(e_{\mathbf{P}}) = e_{\mathbf{M}}$, $\theta(e_{\mathbf{Q}}) = e_{\mathbf{N}}$.

An easy proof of the following proposition is also omitted.

Proposition 2. *For any* $a \in \mathbf{A}$, $b \in \mathbf{B}$, $c \in \mathbf{C}$, $x \in \mathbf{M}$, $y \in \mathbf{L}$:

(1) $\mathbf{E_M}(c) = \tau(c)I$;
(2) $\mathbf{E_A}(b) = \tau(b)I$, $\mathbf{E_B}(a) = \tau(a)I$;
(3) $\tau(ab) = \tau(a)\tau(b)$;
(4) *if* $xa = 0$ *then either* $x = 0$ *or* $a = 0$, *and if* $yb = 0$ *then either* $y = 0$ *or* $b = 0$;
(5) $\mathbf{E_L}(\mathbf{C}) = \mathbf{A}$ *and, therefore,* $\mathbf{E_L}|_{\mathbf{C}} = \mathbf{E_A}$;
(6) *both* $\mathbf{A} \to \overline{\mathbf{A}}$ *and* $\mathbf{B} \to \overline{\mathbf{B}}$ *are isomorphisms.*

Proposition 3. *A map* $\mathbf{A} \otimes \mathbf{A} \to \mathbf{D}$ *defined by* $a_1 \otimes a_2 \mapsto a_1 e_{\mathbf{M}} a_2$ *is a linear isomorphism. Similarly, a map* $\mathbf{B} \otimes \mathbf{B} \to \mathbf{D}$ *defined by* $b_1 \otimes b_2 \mapsto b_1 e_{\mathbf{N}} b_2$ *is a linear isomorphism. This implies that* $\mathbf{D} = \mathbf{A} e_{\mathbf{M}} \mathbf{A} = \mathbf{B} e_{\mathbf{N}} \mathbf{B}$ *and* $\dim \mathbf{A} = \dim \mathbf{B} = d$.

Proof. The map $\mathbf{A} \otimes \mathbf{A} \to \mathbf{D}$ has a trivial kernel. Indeed, if $\{a_i\}$ form a τ-orthonormal basis of $\mathbf{A}$ then $\{a_i e_{\mathbf{M}} a_j^*\}$ are pairwise τ-orthogonal, nonzero vectors and, hence, linearly independent.

The map is onto. To this end it suffices to show that $\mathbf{A}e_{\mathbf{M}}\mathbf{A}$ is an ideal of $\mathbf{C}$. Let $a_1, a_2 \in \mathbf{A}$ and $c \in \mathbf{C}$. By Lemma 1.2 of [6] we have $ca_1e_{\mathbf{M}}a_2 = \lambda^{-1}\mathbf{E_L}(ca_1e_{\mathbf{M}})e_{\mathbf{M}}a_2$ and $\mathbf{E_L}(ca_1e_{\mathbf{M}}) \in \mathbf{A}$ by Proposition 2(5).

With the help of Proposition 1 one can prove in the same way as above that the map $\mathbf{B} \otimes \mathbf{B} \to \mathbf{D}$ is an isomorphism. □

Corollary 4. *For any* $a \in \mathbf{A}$, $b \in \mathbf{B}$, $c \in \mathbf{C}$:

(1) $e_{\mathbf{M}}$ *and* $e_{\mathbf{N}}$ *are minimal projections in* $\mathbf{C}$;

(2) $e_{\mathbf{M}}ae_{\mathbf{M}} = \tau(a)e_{\mathbf{M}}$,
$e_{\mathbf{N}}be_{\mathbf{N}} = \tau(b)e_{\mathbf{N}}$;

(3) $\mathbf{C}e_{\mathbf{M}} = \mathbf{A}e_{\mathbf{M}}$—*more precisely,* $ce_{\mathbf{M}} = \lambda\mathbf{E_A}(ce_{\mathbf{M}})e_{\mathbf{M}}$, *and*
$\mathbf{C}e_{\mathbf{N}} = \mathbf{B}e_{\mathbf{N}}$—*more precisely,* $ce_{\mathbf{N}} = \lambda\mathbf{E_B}(ce_{\mathbf{N}})e_{\mathbf{N}}$; *and*

(4) $\mathbf{D} = \overline{\mathbf{AB}}$ *and* $\mathbf{D} \cong \mathbf{M}_d(\mathbb{C})$, *where* $d = \dim\mathbf{A} = \dim\mathbf{B}$.

Part (1) of the following proposition is established by examining the representation of $\mathbf{L}$ on $\mathbf{L}^2(\mathbf{L}, \tau)$. Part (2) is proven similarly with help of Proposition 1.

Proposition 5. *Let* $\mathbf{D}$ *act on* $\mathscr{H}$ *as a full algebra of endomorphisms. Since* $e_{\mathbf{N}}$ *and* $e_{\mathbf{M}}$ *are minimal in* $\mathbf{D}$ *(Corollary 4(1)), they project onto one-dimensional spaces, say, spanned by* ξ *and* ζ *respectively. Then:*

(1) ζ *is cyclic and separating for* $\overline{\mathbf{A}}$. *If* $J_{\mathbf{A}}$ *denotes the corresponding modular involution then*

(a) $J_{\mathbf{A}}\overline{\mathbf{B}}J_{\mathbf{A}} = \overline{\mathbf{B}}$;

(b) *if* $x, y \in \mathbf{A}$ *then* $J_{\mathbf{A}}(xe_{\mathbf{M}}y)J_{\mathbf{A}} = x^*e_{\mathbf{M}}y^*$; *and*

(c) *if* $x_i, y_i \in \mathbf{A}$ *and* $\sum x_ie_{\mathbf{N}}y_i \in \overline{\mathbf{B}}$ *then* $\sum y_ie_{\mathbf{N}}x_i \in \overline{\mathbf{B}}$.

(2) ξ *is cyclic and separating for* $\overline{\mathbf{B}}$. *If* $J_{\mathbf{B}}$ *denotes the corresponding modular involution then*

(a) $J_{\mathbf{B}}\overline{\mathbf{A}}J_{\mathbf{B}} = \overline{\mathbf{A}}$;

(b) *if* $x, y \in \mathbf{B}$ *then* $J_{\mathbf{B}}(xe_{\mathbf{N}}y)J_{\mathbf{B}} = x^*e_{\mathbf{N}}y^*$; *and*

(c) *if* $x_i, y_i \in \mathbf{B}$ *and* $\sum x_ie_{\mathbf{N}}y_i \in \overline{\mathbf{A}}$ *then* $\sum y_ie_{\mathbf{N}}x_i \in \overline{\mathbf{A}}$.

3. Duality between $\mathbf{N}' \cap \mathbf{L}$ and $\mathbf{M}' \cap \mathbf{K}$

From now on we fix a system of matrix units in $\mathbf{A} \cong \bigoplus_\alpha \mathbf{A}_\alpha$, with each $\mathbf{A}_\alpha$ a factor, as follows:

$$\{s_{ij}^\alpha \mid s_{ii}^\alpha = p_i^\alpha \text{ is a minimal projection in } \mathbf{A}_\alpha \text{ and } s_{ij}^\alpha \text{ is a partial isometry with domain } p_j^\alpha \text{ and range } p_i^\alpha\}.$$

We denote by $/\alpha/$ the natural number such that $\mathbf{A}_\alpha \cong M_{/\alpha/}(\mathbb{C})$.

Definition 6. For any α, i, j we define f_{ij}^α as

$$f_{ij}^\alpha = \tau(p_j^\alpha)^{-1}s_{ji}^\alpha e_{\mathbf{M}}s_{ij}^\alpha.$$

Proposition 7. $\{f_{ij}^\alpha\}$ *are pairwise orthogonal minimal projections in* $\mathbf{D}$ *such that* $\sum_i f_{ij}^\alpha = \overline{p}_j^\alpha$. *Moreover,* $e_{\mathbf{M}}f_{ij}^\alpha = \delta_{ij}e_{\mathbf{M}}p_j^\alpha$, $f_{ij}^\alpha e_{\mathbf{M}} = \delta_{ij}p_j^\alpha e_{\mathbf{M}}$, *and* $s_{ik}^\alpha f_{kk}^\alpha s_{kj}^\alpha = s_{ij}^\alpha f_{kj}^\alpha$.

Proof. Each f_{ij}^{α} is selfadjoint and

$$f_{ij}^{\alpha} f_{mn}^{\beta} = \tau(p_j^{\alpha})^{-1} \tau(p_n^{\beta})^{-1} \tau(s_{ij}^{\alpha} s_{nm}^{\beta}) s_{ji}^{\alpha} e_{\mathbf{M}} s_{mn}^{\beta} = \delta_{\alpha\beta} \delta_{im} \delta_{jn} f_{ij}^{\alpha}.$$

Hence, they are pairwise orthogonal subprojections of $\overline{p}_j^{\alpha}$. Since $\tau(f_{ij}^{\alpha}) = \tau(e_{\mathbf{M}})$, they are minimal in $\mathbf{D}$.

By virtue of Proposition 3 there are scalars t_{mn} such that

$$\overline{p}_j^{\alpha} - \sum_i f_{ij}^{\alpha} = \sum_{m,n} t_{mn} s_{jm}^{\alpha} e_{\mathbf{M}} s_{nj}^{\alpha}.$$

Multiplying it from the left by $s_{jk}^{\alpha} e_{\mathbf{M}} s_{kj}^{\alpha}$ (for an arbitrary k) we get

$$0 = \sum_n t_{kn} s_{jk}^{\alpha} e_{\mathbf{M}} s_{nj}^{\alpha}.$$

Since $s_{jk}^{\alpha} e_{\mathbf{M}} s_{nj}^{\alpha}$ are linearly independent, $t_{kn} = 0$ for all n and $\sum_i f_{ij}^{\alpha} = \overline{p}_j^{\alpha}$.

The remaining claims are easily verified. □

Corollary 8. *For any* α, i *we have* $\tau(\overline{p}_i^{\alpha}) = /\alpha/\lambda^{-1}$.

Since $\tau(p_i^{\alpha}) \geq \tau(\overline{p}_i^{\alpha})$, the preceding corollary is a sharper version of Proposition 1.9 of [6]. It also says that $h(x) := \lambda d^{-1} \tau(\overline{x})$ would be the Haar trace on $\mathbf{A}$ (see Appendix 2 of [8]) if we could define a Hopf algebra structure on $\mathbf{A}$.

Definition 9. We define a bilinear form $(\cdot, \cdot) \colon \mathbf{A} \times \mathbf{B} \to \mathbb{C}$ by

$$(a, b) = \lambda^2 \tau(a e_{\mathbf{M}} e_{\mathbf{N}} b).$$

Proposition 10. *The form defined above establishes duality between* $\mathbf{A}$ *and* $\mathbf{B}$.

Proof. Let $b \in \mathbf{B}$ and, for any $a \in \mathbf{A}$, $(a, b) = 0$. Hence, $0 = \tau(\mathbf{A} e_{\mathbf{M}} e_{\mathbf{N}} b) = \tau(\mathbf{A} e_{\mathbf{M}} \mathbf{A} e_{\mathbf{N}} b) = \tau(e_{\mathbf{N}} \overline{b} \mathbf{D})$. In particular, $0 = \tau(e_{\mathbf{N}} \overline{b}\, \overline{b}^* e_{\mathbf{N}})$ and faithfulness of τ implies $0 = e_{\mathbf{N}} \overline{b}$ and, hence, $b = 0$.

Similarly, $a = 0$ if $(a, b) = 0$ for all $b \in \mathbf{B}$. □

Definition 11. We denote by $\{v_{ij}^{\alpha}\}$ the basis of $\mathbf{B}$ dual to $\{s_{ij}^{\alpha}\}$ viz. $(\cdot, \cdot)$. That is,

$$(s_{kl}^{\beta}, v_{ij}^{\alpha}) = \delta_{\alpha\beta} \delta_{ik} \delta_{jl}.$$

From now on we assume that one of the following easily equivalent conditions is satisfied.

(1) $\mathbf{C}$ is a factor (i.e., $\mathbf{C} = \mathbf{D}$).
(2) $d = \lambda$ ($d = \dim \mathbf{A} = \dim \mathbf{B}$, $\lambda = [\mathbf{M} : \mathbf{N}]$).
(3) $\tau(p_i^{\alpha}) = /\alpha/\lambda^{-1}$ for any α, i.

Proposition 12. (1) $v_{ij}^{\alpha} e_{\mathbf{M}} = \delta_{ij} e_{\mathbf{M}}$.

(2) (a) $(v_{ii}^{\alpha})^* e_{\mathbf{N}} v_{ii}^{\alpha} = /\alpha/^{-1} f_{ii}^{\alpha}$;
(b) $e_{\mathbf{N}} v_{ij}^{\alpha} = e_{\mathbf{N}} v_{jj}^{\alpha} s_{ji}^{\alpha}$, *hence* $/\alpha/^{1/2} e_{\mathbf{N}} v_{ij}^{\alpha}$ *is a partial isometry with domain* f_{ji}^{α} *and range* $e_{\mathbf{N}}$.

(3) (a) $e_{\mathbf{M}} e_{\mathbf{N}} v_{ij}^{\alpha} = /\alpha/^{-1} e_{\mathbf{M}} s_{ji}^{\alpha}$;
(b) $e_{\mathbf{M}} e_{\mathbf{N}} (v_{ij}^{\alpha})^* = /\alpha/^{-1} e_{\mathbf{M}} J_{\mathbf{B}} s_{ji}^{\alpha} J_{\mathbf{B}}$.

(4) $\tau((v_{ij}^{\alpha})^* v_{mn}^{\beta}) = \delta_{\alpha\beta}\delta_{im}\delta_{jn}/\alpha/^{-1}$.
(5) (a) $\sum_k (v_{ik}^{\alpha})^* e_{\mathbf{N}} v_{jk}^{\alpha} = /\alpha/^{-1} s_{ij}^{\alpha}$;
(b) $\sum_k v_{ik}^{\alpha} e_{\mathbf{N}} (v_{jk}^{\alpha})^* = /\alpha/^{-1} J_{\mathbf{B}} s_{ij}^{\alpha} J_{\mathbf{B}}$.
(6) (a) $\mathbf{E_A}((v_{ik}^{\alpha})^* e_{\mathbf{N}} v_{jl}^{\beta}) = \delta_{\alpha\beta}\delta_{kl}/\alpha/^{-2} s_{ij}^{\alpha}$;
(b) $\mathbf{E_A}(v_{ik}^{\alpha} e_{\mathbf{N}} (v_{jl}^{\beta})^*) = \delta_{\alpha\beta}\delta_{kl}/\alpha/^{-2} J_{\mathbf{B}} s_{ij}^{\alpha} J_{\mathbf{B}}$.

Proof. (1) and (2): Since $e_{\mathbf{N}}$ and f_{ii}^{α} are both minimal projections in $\mathbf{D}$, there is a $c \in \mathbf{D}$ such that $c^* e_{\mathbf{N}} c = f_{ii}^{\alpha}$. By Corollary 4(3) there is a $b \in \mathbf{B}$ such that $e_{\mathbf{N}} b = e_{\mathbf{N}} c$; hence, $b^* e_{\mathbf{N}} b = c^* e_{\mathbf{N}} c = f_{ii}^{\alpha}$. Since $e_{\mathbf{M}}$ is minimal and central in $\mathbf{B}$, there is a scalar t such that $b e_{\mathbf{M}} = t e_{\mathbf{M}}$. We have $|t|^2 \lambda^{-1} e_{\mathbf{M}} = |t|^2 e_{\mathbf{M}} e_{\mathbf{N}} e_{\mathbf{M}} = e_{\mathbf{M}}(b^* e_{\mathbf{N}} b) e_{\mathbf{M}} = e_{\mathbf{M}} f_{ii}^{\alpha} e_{\mathbf{M}} = e_{\mathbf{M}} p_i^{\alpha} e_{\mathbf{M}} = \tau(p_i^{\alpha}) e_{\mathbf{M}}$. Therefore, $|t|^2 = \lambda\tau(p_i^{\alpha}) = /\alpha/$ and we can find a $\theta \in \mathbb{R}$ such that $w = e^{i\theta}/\alpha/^{-1/2} b$ satisfies $w^* e_N w = /\alpha/^{-1} f_{ii}^{\alpha}$ and $w e_{\mathbf{M}} = e_{\mathbf{M}}$. Now we have

$$(s_{mn}^{\beta}, w) = \lambda^2 \tau(s_{mn}^{\beta} e_{\mathbf{M}} w^* e_{\mathbf{N}} w) = \tau(p_i^{\alpha})^{-1} \lambda \tau(s_{mn}^{\beta} e_{\mathbf{M}} f_{ii}^{\alpha}) = \delta_{\alpha\beta}\delta_{im}\delta_{in}.$$

Therefore, $w = v_{ii}^{\alpha}$ and we have proven (1) (in case $i = j$) and (2)(a).

Again by Corollary 4(3) there is a $u \in \mathbf{B}$ such that $e_{\mathbf{N}} u = e_{\mathbf{N}} v_{jj}^{\alpha} s_{ji}^{\alpha}$. We have

$$(s_{mn}^{\beta}, u) = \lambda^2 \tau(s_{mn}^{\beta} e_{\mathbf{M}} (v_{jj}^{\alpha})^* e_{\mathbf{N}} v_{jj}^{\alpha} s_{ji}^{\alpha}) = \lambda \tau(p_j^{\alpha})^{-1} \tau(s_{mn}^{\beta} e_{\mathbf{M}} f_{jj}^{\alpha} s_{ji}^{\alpha}) = \delta_{\alpha\beta}\delta_{im}\delta_{jn};$$

hence, $u = v_{ij}^{\alpha}$ and (2)(b) is proven.

If $i \neq j$ then

$$0 = (I, v_{ij}^{\alpha}) = \lambda^2 \tau(e_{\mathbf{M}} e_{\mathbf{N}} v_{ij}^{\alpha}) = \{\text{Proposition 2(3)}\} = \lambda^2 \tau(v_{ij}^{\alpha} e_{\mathbf{M}}) \tau(e_{\mathbf{N}}).$$

This proves (1) in case $i \neq j$.

(3) is established by a direct computation (with help of Proposition 5(2)).

(4)

$$\begin{aligned} \tau((v_{ij}^{\alpha})^* v_{mn}^{\beta}) &= \lambda\tau((v_{ij}^{\alpha})^* e_{\mathbf{N}} v_{mn}^{\beta}) = \lambda\tau(s_{ij}^{\alpha}(v_{jj}^{\alpha})^* e_{\mathbf{N}} v_{nn}^{\beta} s_{nm}^{\beta}) && (2)(b) \\ &= \delta_{\alpha\beta}\delta_{im}\lambda\tau((v_{jj}^{\alpha})^* e_{\mathbf{N}} v_{nn}^{\alpha} s_{nj}^{\alpha}) \\ &= \delta_{\alpha\beta}\delta_{im}\lambda\tau((v_{jj}^{\alpha})^* e_{\mathbf{N}} v_{jn}^{\alpha}) && (2)(b) \\ &= \delta_{\alpha\beta}\delta_{im}\lambda\tau(f_{jj}^{\alpha}(v_{jj}^{\alpha})^* e_{\mathbf{N}} v_{jn}^{\alpha} f_{nj}^{\alpha}) && (2) \\ &= \delta_{\alpha\beta}\delta_{im}\delta_{jn}\lambda\tau((v_{jj}^{\alpha})^* e_{\mathbf{N}} v_{jj}^{\alpha}) = \delta_{\alpha\beta}\delta_{im}\delta_{jn}/\alpha/^{-1}. \end{aligned}$$

(5)(a)

$$\begin{aligned} \sum_k (v_{ik}^{\alpha})^* e_{\mathbf{N}} v_{jk}^{\alpha} &= \sum_k s_{ik}^{\alpha}(v_{kk}^{\alpha})^* e_{\mathbf{N}} v_{kk}^{\alpha} s_{kj}^{\alpha} && (2)(b) \\ &= /\alpha/^{-1} \sum_k s_{ik}^{\alpha} f_{kk}^{\alpha} s_{kj}^{\alpha} && (2)(a) \\ &= /\alpha/^{-1} s_{ij}^{\alpha} \sum_k f_{kj}^{\alpha} = /\alpha/^{-1} s_{ij}^{\alpha}. \end{aligned}$$

(b) follows from (a) with the help of Proposition 5(2)(b).

(6)(a) If $k \neq l$ then $\mathbf{E_A}((v_{kk}^{\alpha})^* e_{\mathbf{N}} v_{ll}^{\alpha}) = 0$. Indeed, for any β, m, n

$$\begin{aligned} \tau(s_{mn}^{\beta}(v_{kk}^{\alpha})^* e_{\mathbf{N}} v_{ll}^{\alpha}) &= \tau(s_{mn}^{\beta} f_{kk}^{\alpha}(v_{kk}^{\alpha})^* e_{\mathbf{N}} v_{ll}^{\alpha} f_{ll}^{\alpha}) && (2)(b) \\ &= \delta_{\alpha\beta}\delta_{kn}\delta_{lm}\tau(f_{ll}^{\alpha} s_{lk}^{\alpha} f_{kk}^{\alpha}(v_{kk}^{\alpha})^* e_{\mathbf{N}} v_{ll}^{\alpha}) \\ &= \delta_{\alpha\beta}\delta_{kn}\delta_{lm}\tau(s_{lk}^{\alpha} f_{lk}^{\alpha} f_{kk}^{\alpha}(v_{kk}^{\alpha})^* e_{\mathbf{N}} v_{ll}^{\alpha}) = 0. \end{aligned}$$

Since clearly $\mathbf{E_A}(f^{\alpha}_{kk}) = /\alpha/^{-1}p^{\alpha}_k$, we obtain

$$\begin{aligned}\mathbf{E_A}((v^{\alpha}_{ik})^*e_{\mathbf{N}}v^{\beta}_{jl}) &= s^{\alpha}_{ik}\mathbf{E_A}((v^{\alpha}_{kk})^*e_{\mathbf{N}}v^{\beta}_{ll})s^{\beta}_{lj} &&(2)(b)\\ &= \delta_{\alpha\beta}\delta_{kl}s^{\alpha}_{ik}\mathbf{E_A}((v^{\alpha}_{kk})^*e_{\mathbf{N}}av^{\alpha}_{kk})s^{\alpha}_{kj}\\ &= \delta_{\alpha\beta}\delta_{kl}/\alpha/^{-1}s^{\alpha}_{ik}\mathbf{E_A}(f^{\alpha}_{kk})s^{\alpha}_{kj} &&(2)(a)\\ &= \delta_{\alpha\beta}\delta_{kl}/\alpha/^{-2}s^{\alpha}_{ij}.\end{aligned}$$

(b) Since $J_{\mathbf{B}}\mathbf{A}J_{\mathbf{B}} = \mathbf{A}$ by virtue of Proposition 4(2)(a) we have $\mathbf{E_A}(J_{\mathbf{B}}xJ_{\mathbf{B}}) = J_{\mathbf{B}}\mathbf{E_A}(x)J_{\mathbf{B}}$ and the claim follows from (a) and Proposition 5(2)(b). □

Corollary 13. *The matrix* $[v^{\alpha}]$, *whose* i, j *entry equals* v^{α}_{ij}, *is a unitary element of* $\mathbf{M}_{/\alpha/}(\mathbf{B})$.

Remark. This fact corresponds to the duality between unitary modules of a Hopf $*$-algebra and unitary comodules of its dual (see [8]).

4. Hopf algebra structures on $\mathbf{N}' \cap \mathbf{L}$ and $\mathbf{M}' \cap \mathbf{K}$

Definition 14. We define linear maps $\Delta\colon \mathbf{B} \to \mathbf{B}\otimes\mathbf{B}$, $\varepsilon\colon \mathbf{B} \to \mathbb{C}$, and $S\colon \mathbf{B} \to \mathbf{B}$ as follows.

(1) $\Delta(b) = \sum b^L_i \otimes b^R_i$, where $\sum b^L_i \otimes b^R_i$ is uniquely determined (by virtue of Proposition 10) by equality $(xy, b) = \sum(x, b^L_i)(y, b^R_i)$, to be satisfied by all $x, y \in \mathbf{A}$.
(2) $\varepsilon(b)$ is defined by $be_{\mathbf{M}} = \varepsilon(b)e_{\mathbf{M}}$.
(3) $S(b)$ is an element of $\mathbf{B}$ uniquely determined (by Proposition 10) by equality $(x, S(b)) = \overline{(x^*, b^*)}$, to be satisfied by all $x \in \mathbf{A}$.

Theorem 15. $\mathbf{B}$ *equipped with its C^*-algebra structure, comultiplication* (Δ), *counit* (ε), *and coinverse* (S) *introduced by Definition* 14 *is a Hopf $*$-algebra and a finite-dimensional compact matrix quantum group as defined in* [8] *by Woronowicz.*

Proof. (1) Since $e_{\mathbf{N}}$ is a minimal and central projection in $\mathbf{A}$, there is a selfadjoint, multiplicative functional $\eta\colon \mathbf{A} \to \mathbb{C}$ such that $ae_{\mathbf{N}} = \eta(a)e_{\mathbf{N}}$ for any $a \in \mathbf{A}$. For any $x, y \in \mathbf{A}$, we have $(xy, I) = \lambda^2\tau(xye_{\mathbf{M}}e_{\mathbf{N}}) = \lambda^2\eta(xy)\tau(e_{\mathbf{N}})\tau(e_{\mathbf{M}}) = \eta(xy)$ and $(x, I)(y, I) = \lambda^4\tau(xe_{\mathbf{M}}e_{\mathbf{N}})\tau(ye_{\mathbf{M}}e_{\mathbf{N}}) = \eta(x)\eta(y)$. Hence, $(xy, I) = (x, I)(y, I)$ and $\Delta(I) = I \otimes I$.

We keep the notation of Proposition 5. We have $\tau(e_{\mathbf{N}}xe_{\mathbf{N}}) = \lambda^{-1}\langle x\xi, \xi\rangle$ for any $x \in \mathbf{D}$; hence, $(a, b) = \lambda\langle ae_{\mathbf{M}}\xi, b^*\xi\rangle$ for any $a \in \mathbf{A}$, $b \in \mathbf{B}$. For any $x, y \in \mathbf{A}$ we have

$$\begin{aligned}(xy, b^*) &= \lambda\langle xye_{\mathbf{M}}\xi, b\xi\rangle\\ &= \lambda\overline{\langle (J_{\mathbf{B}}xJ_{\mathbf{B}})(J_{\mathbf{B}}yJ_{\mathbf{B}})e_{\mathbf{M}}\xi, b^*\xi\rangle}\\ &= \lambda^2\sum\overline{\langle J_{\mathbf{B}}xJ_{\mathbf{B}}e_{\mathbf{M}}\xi, (b^L_i)^*\xi\rangle}\;\overline{\langle J_{\mathbf{B}}yJ_{\mathbf{B}}e_{\mathbf{M}}\xi, (b^R_i)^*\xi\rangle}\\ &= \sum(x, (b^L_i)^*)(y, (b^R_i)^*).\end{aligned}$$

This proves that $\Delta(b^*) = \Delta(b)^*$.

(2) Associativity of the multiplication in $\mathbf{A}$ implies $(\Delta\otimes\mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta$.

(3) Since $e_{\mathbf{M}}$ is a minimal and central projection in $\mathbf{B}$, ε is a selfadjoint, multiplicative functional.

(4) For any $c \in \mathbf{B}$ we have $(I, c) = \lambda^2\tau(e_{\mathbf{M}}e_{\mathbf{N}}c) = \varepsilon(c)$. Therefore, for any $a \in \mathbf{A}$, $(a, b) = (a \cdot I, b) = \sum(a, b_i^L)(I, b_i^R) = (a, \sum \varepsilon(b_i^R)b_i^L)$; hence, $b = \sum \varepsilon(b_i^R)b_i^L$ and $(\mathrm{id} \otimes \varepsilon)\Delta = \mathrm{id}$. Similarly, $(\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}$.

(5) Let $b \in \mathbf{B}$ and $b = \sum x_i e_M y_i$ with $x_i, y_i \in \mathbf{A}$. By Proposition 5(1), $\sum y_i e_{\mathbf{M}} x_i = J_{\mathbf{A}}(\sum y_i^* e_{\mathbf{M}} x_i^*)J_{\mathbf{A}}$ belongs to $\mathbf{B}$. Since

$$\left(a, \sum y_i e_{\mathbf{M}} x_i^*\right) = \left(\overline{a^*, \sum y_i^* e_{\mathbf{M}} x_i^*}\right)$$

for any $a \in \mathbf{A}$, $S(b) = J_{\mathbf{A}} b^* J_{\mathbf{A}}$ for any $b \in \mathbf{B}$. This implies that S is a $*$-preserving, antimultiplicative involution.

(6) We claim that $S(v_{ij}^\alpha) = (v_{ji}^\alpha)^*$. Indeed, by Proposition 12(3) we have

$$(a, (v_{ji}^\alpha)^*) = \lambda^2\tau(ae_{\mathbf{M}}e_{\mathbf{N}}(v_{ji}^\alpha)^*) = /\alpha/^{-1}\lambda^2\tau(ae_{\mathbf{M}}J_{\mathbf{B}}s_{ij}^\alpha J_{\mathbf{B}}) = /\alpha/^{-1}\lambda\tau(J_{\mathbf{B}}s_{ij}^\alpha J_{\mathbf{B}}a).$$

Similarly,

$$\overline{(a^*, (v_{ij}^\alpha)^*)} = \lambda^2\tau(v_{ij}^\alpha e_{\mathbf{N}}e_{\mathbf{M}}a) = /\alpha/^{-1}\lambda^2\tau(J_{\mathbf{B}}s_{ij}^\alpha J_{\mathbf{B}}e_{\mathbf{M}}a) = /\alpha/^{-1}\lambda\tau(aJ_{\mathbf{B}}s_{ij}^\alpha J_{\mathbf{B}}),$$

and the claim follows. Since it is an immediate consequence of our definitions that $\Delta(v_{ij}^\alpha) = \sum_k v_{ik}^\alpha \otimes v_{kj}^\alpha$, it now follows easily from Proposition 12(1) and Corollary 13 that $m(S \otimes \mathrm{id})\Delta = \varepsilon = m(\mathrm{id} \otimes S)\Delta$, where $m\colon b_1 \otimes b_2 \mapsto b_1 b_2$.

(7) Now we are in a position to prove multiplicativity of Δ.

Let us define a map $T\colon \mathbf{B} \otimes \mathbf{B} \to \mathbf{D}$ by $T\colon b_1 \otimes b_2 \mapsto b_1 e_{\mathbf{N}} S(b_2)$. By Proposition 3 and (5) this map is a linear isomorphism. We claim that $T(\Delta(\mathbf{B})) = \mathbf{A}$.

At first we notice that (4) implies that Δ is an injective map. We have

$$T(\Delta(v_{ij}^\alpha)) = \sum_k v_{ik}^\alpha e_{\mathbf{N}} S(v_{kj}^\alpha) = \sum_k v_{ik}^\alpha e_{\mathbf{N}} (v_{jk}^\alpha)^*$$

belongs to $\mathbf{A}$ by Propositions 5(2) and 12(5). Since $\{\Delta(v_{ij}^\alpha)\}$ form a linear basis of $\Delta(\mathbf{B})$, we see that $T(\Delta(\mathbf{B})) \subseteq \mathbf{A}$, and, since T is injective and $\dim \Delta(\mathbf{B}) = \dim \mathbf{A}$, we conclude that $T(\Delta(\mathbf{B})) = \mathbf{A}$.

Now we define a new multiplication $\odot$ in $\mathbf{D}$ by $x \odot y = T(T^{-1}(x)T^{-1}(y))$. One can easily check that if $b_1, b_2, c_1, c_2 \in \mathbf{B}$ then $(b_1 e_{\mathbf{N}} c_1) \odot (b_2 e_{\mathbf{N}} c_2) = b_1 b_2 e_{\mathbf{N}} c_2 c_1$ (by (5) S is an antimultipicative involution).

Since Δ has a multiplicative inverse $\varepsilon \otimes \mathrm{id}$, in order to prove multiplicativity of Δ it suffices to show that $\Delta(\mathbf{B})$ is closed in $\mathbf{B} \otimes \mathbf{B}$ under multiplication or, equivalently, that $\mathbf{A}$ is closed in $\mathbf{D}$ under $\odot$. We have

$$T(\Delta(v_{ij}^\alpha)) \odot T(\Delta(v_{mn}^\beta)) = \sum_k v_{ik}^\alpha \left(\sum_t v_{mt}^\beta e_{\mathbf{N}} (v_{nt}^\beta)^*\right) (v_{jk}^\alpha)^*$$

and $\sum_t v_{mt}^\beta e_{\mathbf{N}} (v_{nt}^\beta)^*$ is in $\mathbf{A}$. We have to show that $\sum_k v_{ik}^\alpha a (v_{jk}^\alpha)^*$ is in $\mathbf{A}$ for any $a \in \mathbf{A}$. Since $\mathbf{A} = \mathbf{N}' \cap \mathbf{L}$ and $\sum_k v_{ik}^\alpha a (v_{jk}^\alpha)^*$ being in $\mathbf{D}$ is in $\mathbf{N}'$, it remains to show that $\sum_k v_{ik}^\alpha a (v_{jk}^\alpha)^*$ is in $\mathbf{L}$. But there are elements x_s, y_s in $\mathbf{M}$ such that $a = \sum x_s e_{\mathbf{N}} y_s$. Therefore,

$$\sum_k v_{ik}^\alpha a (v_{jk}^\alpha)^* = \sum_k v_{ik}^\alpha \left(\sum_s x_s e_{\mathbf{N}} y_s\right) (v_{jk}^\alpha)^* = \sum_s x_s \left(\sum_k v_{ik}^\alpha e_{\mathbf{N}} (v_{jk}^\alpha)^*\right) y_s,$$

and, since $\sum_k v_{ik}^\alpha e_{\mathbf{N}} (v_{jk}^\alpha)^*$ belongs to $\mathbf{A} \subseteq \mathbf{L}$, we are done.

(8) One can check that a matrix with blocks $[v^\alpha]$ along its diagonal satisfies the requirements of the definition of a compact matrix quantum group. □

Corollary 16. $\mathbf{A}$ *with its natural* C^**-algebra structure, comultiplication, counit, and antipode defined as dual to multiplication, unit, and antipode respectively in* $\mathbf{B}$ (*viz.* $(\cdot,\cdot)$) *bears a Hopf* $*$*-algebra structure dual to that of* $\mathbf{B}$.

5. Action of $\mathbf{M}' \cap \mathbf{K}$ on $\mathbf{L}$

Definition 17. We define a bilinear map $\mathbf{B}\times\mathbf{L}\to\mathbf{L}$ (denoted by $b \rightharpoonup x$) by setting $b \rightharpoonup x = \lambda \mathbf{E_L}(bxe_\mathbf{M})$.

Lemma 18. *For any* $b\in\mathbf{B}$, $x\in\mathbf{L}$ *we have* $bx = \sum((b_i^L)\rightharpoonup x)b_i^R$.

Proof. Since $\{v_{ij}^\alpha\}$ form a basis of $\mathbf{B}$ and $\Delta(v_{ij}^\alpha)=\sum_k v_{ik}^\alpha\otimes v_{kj}^\alpha$, it suffices to show that $v_{ij}^\alpha x = \lambda\sum_k \mathbf{E_L}(v_{ik}^\alpha x e_\mathbf{M})v_{kj}^\alpha$. At first we prove this for $x=e_\mathbf{N}$.

Since $\dim\mathbf{B}=\lambda$ and $\mathbf{E_L}|_\mathbf{B}=\tau$, we infer that $\mathbf{B}$ contains a quasi-basis for $\mathbf{E_L}$ (see [6, 7]). It follows that there are elements $x_{ns}^\gamma\in\mathbf{L}$ such that

$$v_{ij}^\alpha e_\mathbf{N} = \sum_{\gamma ns} x_{ns}^\gamma v_{ns}^\gamma.$$

Multiplying this equality by $(v_{km}^\beta)^*$ (for any fixed β, k, m) from the right and taking $\mathbf{E_L}$ of both sides we get

$$\begin{aligned}\mathbf{E_L}(v_{ij}^\alpha e_\mathbf{N}(v_{km}^\beta)^*) &= \sum_{\gamma ns} x_{ns}^\gamma \mathbf{E_L}(v_{ns}^\gamma(v_{km}^\beta)^*)\\ &= \sum_{\gamma ns} x_{ns}^\gamma \tau(v_{ns}^\gamma(v_{km}^\beta)^*) = /\beta/^{-1}x_{km}^\beta.\end{aligned}$$

Since $\mathbf{E_L}|_\mathbf{D}=\mathbf{E_A}$, by virtue of Proposition 12(6)(b) we have

$$\begin{aligned}v_{ij}^\alpha e_\mathbf{N} &= \sum_{\beta km} /\beta/\mathbf{E_A}(v_{ij}^\alpha e_\mathbf{N}(v_{km}^\beta)^*)v_{km}^\beta\\ &= /\alpha/^{-1}\sum_k (J_\mathbf{B} s_{ik}^\alpha J_\mathbf{B})v_{kj}^\alpha.\end{aligned}$$

On the other hand, with the help of Propositions 5(2) and 12(3) we get

$$\mathbf{E_A}(v_{ik}^\alpha e_\mathbf{N} e_\mathbf{M}) = /\alpha/^{-1}\mathbf{E_A}((J_\mathbf{B} s_{ik}^\alpha J_\mathbf{B})e_\mathbf{M}) = \lambda^{-1}/\alpha/^{-1}J_\mathbf{B} s_{ik}^\alpha J_\mathbf{B}.$$

Thus the lemma is proven for $x=e_\mathbf{N}$. The general case follows from the fact that for any $x\in\mathbf{L}$ there are $y_n, t_n\in\mathbf{M}$ with $x=\sum y_n e_\mathbf{N} t_n$ (cf. [3, Theorem 3.6.4(iii)]). □

Proposition 19. *The map introduced by Definition* 17 *is a left action of* $\mathbf{B}$ *on* $\mathbf{L}$.

Proof. (1) $I\rightharpoonup x = \lambda\mathbf{E_L}(xe_\mathbf{M}) = \lambda x\mathbf{E_L}(e_\mathbf{M}) = x$.

(2) $b\rightharpoonup I = \lambda\mathbf{E_L}(be_\mathbf{M}) = \varepsilon(b)I$.

(3) By virtue of Lemma 1.2 of [6] we have

$$\begin{aligned}b_1b_2\rightharpoonup x &= \lambda\mathbf{E_L}(b_1b_2xe_\mathbf{M})\\ &= \lambda\mathbf{E_L}(b_1\lambda\mathbf{E_L}(b_2xe_\mathbf{M})e_\mathbf{M}) = b_1\rightharpoonup(b_2\rightharpoonup x).\end{aligned}$$

(4) Proposition 12(3) implies that $\mathbf{E_L}(e_\mathbf{M} e_\mathbf{N} b) = \mathbf{E_L}(S(b) e_\mathbf{N} e_\mathbf{M})$ for any $b \in \mathbf{B}$. Since any $x \in \mathbf{L}$ can be written as $x = \sum u_i e_\mathbf{N} w_i$, $u_i, w_i \in \mathbf{M}$, the above gives $\mathbf{E_L}(e_\mathbf{M} x b) = \mathbf{E_L}(S(b) x e_\mathbf{M})$. Thus, for any $b \in \mathbf{B}$, $x \in \mathbf{L}$, $(b \rightharpoonup x)^* = \lambda \mathbf{E_L}(e_\mathbf{M} x^* b^*) = \lambda \mathbf{E_L}(S(b^*) x^* e_\mathbf{M}) = S(b^*) \rightharpoonup x$.

(5) By the preceding lemma $bx = \lambda \sum \mathbf{E_L}(b_i^L x e_\mathbf{M}) b_i^R$ for any $b \in \mathbf{B}$, $x \in \mathbf{L}$. Multiplying this equality from the right by $y e_\mathbf{M} t$ with $y, t \in \mathbf{L}$ arbitrary and then taking trace of both sides we get $\tau(bxye_\mathbf{M} t) = \lambda \tau(\sum \mathbf{E_L}(b_i^L x e_\mathbf{M}) b_i^R y e_\mathbf{M} t)$. This yields

$$\mathbf{E_L}(bxye_\mathbf{M}) = \lambda \sum \mathbf{E_L}(b_i^L x e_\mathbf{M}) \mathbf{E_L}(b_i^R y e_\mathbf{M}).$$

This means that $b \rightharpoonup (xy) = \sum (b_i^L \rightharpoonup x)(b_i^R \rightharpoonup y)$. □

At this point we have no difficulties in proving our final result.

Theorem 20. *Let* $\mathbf{B}$ *act on* $\mathbf{L}$ *from the left as in Definition* 17. *A map* ϕ *such that* $\phi\colon x \otimes b \mapsto xb$, $x \in \mathbf{L}$, $b \in \mathbf{B}$, *is a* $*$*-isomorphism from the crossed product* $\mathbf{L} \rtimes \mathbf{B}$ *onto* $\mathbf{K}$.

Proof. If $\{b_i\}$ form a τ-orthonormmal basis of $\mathbf{B}$ then $\{(b_i, b_i^*)\}$ form a quasi basis for $\mathbf{E_L}$ (cf. [7; 6, Proposition 1.3]). Thus the map is a linear isomorphism. We have $\phi(I \otimes I) = I$ and for any $x \in \mathbf{L}$, $b \in \mathbf{B}$

$$\begin{aligned}\phi((x \otimes b)^*) &= \phi\left(\sum ((b_i^L)^* \rightharpoonup x^*) \otimes (b_i^R)^*\right) \\ &= \sum ((b_i^L)^* \rightharpoonup x^*)(b_i^R)^*.\end{aligned}$$

On the other hand, since Δ preserves $*$, applying Lemma 18 we get

$$\phi(x \otimes b)^* = b^* x^* = \sum ((b_i^L)^* \rightharpoonup x^*)(b_i^R)^*,$$

thus ϕ preserves $*$. ϕ also preserves multiplication, since for any $x_1, x_2 \in \mathbf{L}$, $b_1, b_2 \in \mathbf{B}$ we have (again by Lemma 18)

$$\begin{aligned}\phi((x_1 \otimes b_1)(x_2 \otimes b_2)) &= x_1 \left(\sum (b_1^L \rightharpoonup x_2) b_1^R\right) b_2 \\ &= x_1 b_1 x_2 b_2 = \phi(x_1 \otimes b_1)\phi(x_2 \otimes b_2). \quad \square\end{aligned}$$

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Department of Mathematical Sciences, Old Chemistry Bld., ML 25, University of Cincinnati, Cincinnati, Ohio 45221

E-mail address: `szymanski@ucbeh.san.uc.edu`