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THE SET OF ALL $m \times n$ RECTANGULAR REAL MATRICES OF RANK r IS CONNECTED BY ANALYTIC REGULAR ARCS

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ABSTRACT. It is well known that the set of all square invertible real matrices has two connected components. The set of all $m \times n$ rectangular real matrices of rank r has only one connected component when $m \neq n$ or $r < m = n$. We show that all these connected components are connected by analytic regular arcs. We apply this result to establish the existence of p-times differentiable bases of the kernel and the image of a rectangular real matrix function of several real variables.

INTRODUCTION

In [3] we showed that every open connected subset of a topological vector space is connected by regular polynomial curves. In this paper, we deal with the set of real $m \times n$ matrices of rank r. In spite of the fact that this set is not an open subset of $\mathbb{R}^{m \times n}$, we show that it is connected by regular arcs that are not only of class C^∞ but are also analytic. A similar result was established in [2, Theorem 7.2] for complex matrices, but some new methods are necessary to obtain arcs that are contained in the set of real matrices. We furnish a method to construct these arcs explicitly.

The analytic connections are likely to have many applications. For example, a method to construct continuous arcs in the set of square invertible real matrices is furnished in [1, Proposition 1.5]. This construction was used to establish the uniqueness of the topological degree. In this paper, we provide another application by showing that the main result of [2] about the existence of bases of class C^p of the kernel and the image of a rectangular matrix function of several real variables is also valid when the field is $\mathbb{R}$ instead of $\mathbb{C}$.

We will denote by $\mathbb{R}^{m \times n}$ the set of all $m \times n$ real matrices and by $\mathbb{R}_r^{m \times n}$ the subset of $\mathbb{R}^{m \times n}$ of all matrices of rank r. We will denote by I_n the $n \times n$ identity matrix and by $I_r^{m \times n}$ the following $m \times n$ matrix of rank r:

$$I_r^{m \times n} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}.$$

In Lemma 1 we show that the two connected components of $\mathbb{R}_n^{n \times n}$ are connected by analytic arcs that may be chosen as closed curves travelled infinitely

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many times. In Lemma 2 we establish the existence of equivalences with positive determinants between matrices of same rank. In Lemma 3 we show that $\mathbb{R}_r^{m\times n}$ is connected by analytic arcs that may be chosen as closed curves travelled infinitely many times, when $m \neq n$ or $r < m = n$. In Theorem 4 we show that all the connected components of $\mathbb{R}_n^{n\times n}$ and $\mathbb{R}_r^{m\times n}$ are connected by analytic arcs that are regular. The problem of finding analytic regular closed curves travelled infinitely many times is still open. In Theorem 5 we apply Theorem 4 to establish the existence of bases of class C^p of the kernel and the image of a rectangular real matrix function of several real variables.

RESULTS

Lemma 1. *Let A, $B \in \mathbb{R}_n^{n\times n}$ be such that $\det A$ and $\det B$ have the same sign. Then there exists an analytic mapping $F: \mathbb{R} \to \mathbb{R}_n^{n\times n}$ such that, for every $m \in \mathbb{Z}$, $F(m) = A$ if m is even and $F(m) = B$ if m is odd. Moreover, $\det F(t)$ has the same sign as $\det A$, and $F(t+2) = F(t)$ for every $t \in \mathbb{R}$.*

Proof. Let $C = BA^{-1} \in \mathbb{R}_n^{n\times n}$. Since $\det A$ and $\det B$ have the same sign, we have $\det C > 0$. It is well known that C is similar in $\mathbb{R}^{n\times n}$ to a real Jordan matrix. More precisely, there exist $R \in \mathbb{R}_n^{n\times n}$ and $J \in \mathbb{R}^{n\times n}$ such that $C = RJR^{-1}$, where J has the form

$$J = \mathrm{diag}[J_1, J_2, J_3],$$

where in turn J_1 has the form

$$J_1 = \begin{bmatrix} C(\rho_1, \theta_1) & \alpha_1 I_2 & & 0 \\ & & \ddots & \\ & & \ddots & \alpha_{p-1} I_2 \\ 0 & & & C(\rho_p, \theta_p) \end{bmatrix},$$

$\rho_1, \dots, \rho_p > 0$, $\theta_1, \dots, \theta_p \in [0, 2\pi[$, $\alpha_1, \dots, \alpha_{p-1} \in \{0, 1\}$,

$$C(\rho, \theta) = \rho \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad \forall \rho > 0, \ \theta \in [0, 2\pi[;$$

J_2 has the form

$$J_2 = \begin{bmatrix} \mu_1 & \beta_1 & & 0 \\ & & \ddots & \\ & & \ddots & \beta_{q-1} \\ 0 & & & \mu_q \end{bmatrix},$$

$\mu_1, \dots, \mu_q > 0$, $\beta_1, \dots, \beta_{q-1} \in \{0, 1\}$; and J_3 has the form

$$J_3 = \begin{bmatrix} v_1 & \gamma_1 & & 0 \\ & \cdot & \ddots & \\ & & \ddots & \gamma_{r-1} \\ 0 & & & v_r \end{bmatrix},$$

$v_1, \dots, v_r < 0$, $\gamma_1, \dots, \gamma_{r-1} \in \{0, 1\}$, and r is even because $\det C > 0$. Let

$$\lambda(t) = \cos^2(\pi t/2), \qquad \mu(t) = \sin^2(\pi t/2) \quad \forall t \in \mathbb{R}.$$

Let $t \in \mathbb{R}$. For every $k \in \{1, 2, \dots\}$ that makes sense, let

$$r_k(t) = \lambda(t) + \mu(t)\rho_k, \qquad s_k(t) = \mu(t)\theta_k, \qquad a_k(t) = \mu(t)\alpha_k,$$
$$C_k(t) = C(r_k(t), s_k(t)),$$

$$m_k(t) = \lambda(t) + \mu(t)\mu_k, \qquad b_k(t) = \mu(t)\beta_k,$$
$$n_k(t) = -\lambda(t) + \mu(t)v_k, \qquad c_k(t) = \mu(t)\gamma_k.$$

Let

$$H_1(t) = \begin{bmatrix} C_1(t) & a_1(t)I_2 & & 0 \\ & & \ddots & \\ & \ddots & & a_{p-1}(t)I_2 \\ 0 & & & C_p(t) \end{bmatrix},$$

$$H_2(t) = \begin{bmatrix} m_1(t) & b_1(t) & & 0 \\ & & \ddots & \\ & \ddots & & b_{q-1}(t) \\ 0 & & & m_q(t) \end{bmatrix},$$

$$R(t) = \begin{bmatrix} \cos(\pi t) & \sin(\pi t) \\ -\sin(\pi t) & \cos(\pi t) \end{bmatrix},$$

$$H_3(t) = \begin{bmatrix} n_1(t) & c_1(t) & & 0 \\ & & \ddots & \\ & \ddots & & c_{r-1}(t) \\ 0 & & & n_r(t) \end{bmatrix} \begin{bmatrix} -R(t) & & 0 \\ & \ddots & \\ 0 & & -R(t) \end{bmatrix},$$

where the last matrix has $\frac{r}{2}$ blocks $R(t)$, which is possible since r is even.

$$H(t) = \operatorname{diag}[H_1(t), H_2(t), H_3(t)],$$
$$G(t) = RH(t)R^{-1}, \qquad F(t) = G(t)A.$$

Let $m \in \mathbb{Z}$. Plainly, if m is even, then

$$H_1(m) = I_{2p}, \quad H_2(m) = I_q, \quad H_3(m) = (-I_r)(-I_r) = I_r,$$
$$H(m) = I_n, \quad G(m) = RR^{-1} = I_n, \quad F(m) = I_nA = A,$$

and, if m is odd, then

$$H_1(m) = J_1, \quad H_2(m) = J_2, \quad H_3(m) = J_3,$$
$$H(m) = J, \quad G(m) = RJR^{-1} = C, \quad F(m) = CA = B.$$

It is obvious that all the functions above are periodic with period 2. Let $t \in \mathbb{R}$. It is easy to see that

$$\det H_1(t) = \det C_1(t) \cdots \det C_p(t) = (r_1(t))^2 \cdots (r_p(t))^2 > 0,$$
$$\det H_2(t) = m_1(t) \cdots m_q(t) > 0,$$
$$\det H_3(t) = n_1(t) \cdots n_r(t)(\det R(t))^{r/2} = (-1)^r |n_1(t)| \cdots |n_r(t)| > 0,$$

because r is even, and, finally,

$$\det H(t) = (\det H_1(t))(\det H_2(t))(\det H_3(t)) > 0,$$
$$\det(G(t)) = \det H(t) > 0,$$
$$\det F(t) = (\det G(t))(\det A) \neq 0.$$

Thus $F(t) \in \mathbb{R}_n^{n\times n}$, and $\det F(t)$ has the same sign as $\det A$. □

Lemma 2. *Let m, $n \in \{1, 2, \dots\}$ and $r \in \{0, 1, 2, \dots\}$ be such that $m \neq n$ or $r < m = n$. Let $A \in \mathbb{R}_r^{m\times n}$. Then there exist $L \in \mathbb{R}_m^{m\times m}$ and $R \in \mathbb{R}_n^{n\times n}$ such that $\det L > 0$, $\det R > 0$, and $A = LI_r^{m\times n}R$.*

Proof. If $r = 0$, then $A = 0 = I_r^{m\times n}$, and we can choose $L = I_m$ and $R = I_n$. Suppose $r \geq 1$. It follows from the hypothesis that $r < m$ or $r < n$.

(a) Suppose $r < m$. Then $m \geq 2$. As A is of rank r, it is well known that A is equivalent in $\mathbb{R}_r^{m\times n}$ to $I_r^{m\times n}$. That is, there exist $B \in \mathbb{R}_m^{m\times m}$ and $C \in \mathbb{R}_n^{n\times n}$ such that $A = BI_r^{m\times n}C$. If $\det B > 0$ and $\det C > 0$, we obviously choose $L = B$ and $R = C$. If $\det B > 0$ and $\det C < 0$, we choose

$$L = B\,\mathrm{diag}[-1, I_{m-2}, -1] \quad \text{and} \quad R = \mathrm{diag}[-1, I_{n-1}]C,$$

considering that $r = n$ is possible. If $\det B < 0$ and $\det C > 0$, we choose

$$L = B\,\mathrm{diag}[I_{m-1}, -1] \quad \text{and} \quad R = C.$$

If $\det B < 0$ and $\det C < 0$, we choose

$$L = B\,\mathrm{diag}[-1, I_{m-1}] \quad \text{and} \quad R = \mathrm{diag}[-1, I_{n-1}]C.$$

It is easy to check that, in all cases, $A = LI_r^{m\times n}R$, $\det L > 0$, and $\det R > 0$.

(b) Suppose $r < n$. Then by (a), there exist $B \in \mathbb{R}_n^{n\times n}$ and $C \in \mathbb{R}_m^{m\times m}$ such that $A^\top = BI_r^{n\times m}C$, $\det B > 0$, and $\det C > 0$. Let $L = C^\top$ and $R = B^\top$. Then

$$A = C^\top I_r^{m\times n}B^\top = LI_r^{m\times n}R,$$
$$\det L = \det C^\top = \det C > 0, \qquad \det R = \det B^\top = \det B > 0. \quad \square$$

Lemma 3. *Let m, $n \in \{1, 2, \dots\}$ and $r \in \{0, 1, \dots\}$ be such that $m \neq n$ or $r < m = n$. Let A, $B \in \mathbb{R}_r^{m\times n}$. Then there exists an analytic mapping $F: \mathbb{R} \to \mathbb{R}_r^{m\times n}$ such that, for every $k \in \mathbb{Z}$, $F(k) = A$ if k is even and $F(k) = B$ if k is odd. Moreover, $F(t+2) = F(t)$ for every $t \in \mathbb{R}$.*

Proof. By Lemma 2 there exist A_1, $B_1 \in \mathbb{R}_m^{m\times m}$ and A_2, $B_2 \in \mathbb{R}_n^{n\times n}$ such that

$$A = A_1 I_r^{m\times n} A_2, \qquad B = B_1 I_r^{m\times n} B_2,$$
$$\det A_1 > 0, \qquad \det A_2 > 0, \qquad \det B_1 > 0, \qquad \det B_2 > 0.$$

By Lemma 1 there exist analytic mappings $F_1: \mathbb{R} \to \mathbb{R}_m^{m\times m}$ and $F_2: \mathbb{R} \to \mathbb{R}_n^{n\times n}$ such that, for every $k \in \mathbb{Z}$, $F_1(k) = A_1$, $F_2(k) = A_2$ if k is even and $F_1(k) = B_1$, $F_2(k) = B_2$ if k is odd. Let

$$F(t) = F_1(t) I_r^{m\times n} F_2(t) \quad \forall t \in \mathbb{R}.$$

Then $F: \mathbb{R} \to \mathbb{R}_r^{m\times n}$ is analytic, and, for every $k \in \mathbb{Z}$, $F(k) = A_1 I_r^{m\times n} A_2 = A$ if k is even and $F(k) = B_1 I_r^{m\times n} B_2 = B$ if k is odd. □

Theorem 4. *The subset $\mathbb{R}_n^{n\times n}$ of $\mathbb{R}^{n\times n}$ has two connected components, whereas the subset $\mathbb{R}_r^{m\times n}$ of $\mathbb{R}^{m\times n}$ has only one connected component when $m \neq n$ or*

$r < m = n$. When $r > 0$, all these connected components are connected by analytic regular arcs. More precisely: Suppose $r > 0$. Let A, $B \in \mathbb{R}_r^{m\times n}$ be such that $A \neq B$, and, if $r = m = n$, then $\det A$ and $\det B$ have the same sign. Then there exists an analytic mapping $F: \mathbb{R} \to \mathbb{R}_r^{m\times n}$ such that $F(0) = A$, $F(1) = B$, and $F'(t) \neq 0$ for every $t \in [0, 1]$.

Proof. By Lemma 1 (if $r = m = n$) and Lemma 3 (if $m \neq n$ or $r < m = n$), there exists an analytic mapping $G: \mathbb{R} \to \mathbb{R}_r^{m\times n}$ such that $G(0) = A$ and $G(1) = B$. Let

$$\chi = \{t \in [0, 1] \mid \exists \lambda(t) \in \mathbb{R},\ G'(t) = \lambda(t)G(t)\}.$$

Case 1: Suppose χ is infinite. Then there exist $t_1, t_2, \ldots \in \chi$ and $t_0 \in [0, 1]$ such that $t_0 = \lim_{k\to\infty} t_k$. For every $t \in \mathbb{R}$, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$, let $g_{ij}(t)$ denote the entry of the ith row, jth column of $G(t)$. Since $r > 0$, there exist $i_0 \in \{1, \ldots, m\}$ and $j_0 \in \{1, \ldots, n\}$ such that $g_{i_0j_0}(t_0) \neq 0$. As $g_{i_0j_0}$ is continuous, there exists a neighborhood $N_{t_0} \subseteq \mathbb{R}$ of t_0 such that $g_{i_0j_0}(t) \neq 0$ for every $t \in N_{t_0}$. Because $\lim_{k\to\infty} t_k = t_0$, there exists $k_0 \in \mathbb{N}$ such that $t_k \in N_{t_0}$ for every $k \in \{k_0, k_0+1, \ldots\}$. Let $k \in \{k_0, k_0+1, \ldots\}$. Since $t_k \in \chi$, we have $G'(t_k) = \lambda(t_k)G(t_k)$, which implies $g'_{i_0j_0}(t_k) = \lambda(t_k)g_{i_0j_0}(t_k)$ and, hence, $g_{i_0j_0}(t_k)G'(t_k) = g'_{i_0j_0}(t_k)G(t_k)$. As $\lim_{k\to\infty} t_k = t_0$, it follows by the Analytic Continuation Theorem that $g_{i_0j_0}(t)G'(t) = g'_{i_0j_0}(t)G(t)$ for every $t \in \mathbb{R}$. Let $g = g_{i_0j_0}$. The equality $gG' - g'G = 0$ implies that $(G(t)/g(t))' = 0$ for every $t \in N_{t_0}$. Therefore, there exists a constant matrix $M_0 \in \mathbb{R}^{m\times n}$ such that $G(t) = g(t)M_0$ for every $t \in N_{t_0}$ and hence for every $t \in \mathbb{R}$ by analytic continuation. Because $\operatorname{rank} G = r > 0$, the equality $G = gM_0$ implies that $g(t) \neq 0$ for every $t \in \mathbb{R}$ and $M_0 \neq 0$. Furthermore, $A = G(0) = g(0)M_0$ and $B = G(1) = g(1)M_0$. Let

$$F(t) = \{t(g(1) - g(0)) + g(0)\}M_0 \quad \forall t \in \mathbb{R}.$$

Then

$$F(0) = g(0)M_0 = A, \qquad F(1) = g(1)M_0 = B,$$

and

$$F'(t) = (g(1) - g(0))M_0 \neq 0,$$

for every $t \in \mathbb{R}$, because $A \neq B$ implies that $g(0) \neq g(1)$.

Case 2: Suppose that χ is finite. Then there exist $t_0, \ldots, t_q \in [0, 1]$ such that $0 = t_0 < t_1 < \cdots < t_q = 1$ and

$$\chi \subseteq \{t_0, \ldots, t_q\}. \tag{1}$$

By Hermite interpolation, there exists a polynomial $p \in \mathbb{R}[X]$ such that, for every $k \in \{0, \ldots, q\}$,

$$p(t_k) = \tfrac{1}{2}, \tag{2}$$

$$t_k \in \chi \text{ and } \lambda(t_k) \neq 0 \Rightarrow p'(t_k) = \lambda(t_k), \tag{3}$$

$$t_k \in \chi \text{ and } \lambda(t_k) = 0 \Rightarrow p'(t_k) = 1. \tag{4}$$

For every $t \in \mathbb{R}$, let

$$q(t) = p(t)^2 + \tfrac{3}{4}, \qquad F(t) = q(t)G(t).$$

Then, by (2), $F(0) = G(0) = A$ and $F(1) = G(1) = B$. Moreover, for every $t \in \mathbb{R}$, we have $\operatorname{rank} F(t) = \operatorname{rank} G(t) = r$, because $q(t) \neq 0$. Let $t \in [0, 1]$. Let us show that $F'(t) \neq 0$. Suppose $F'(t) = 0$. Then

$$0 = F'(t) = q'(t)G(t) + q(t)G'(t). \tag{5}$$

Consequently,

$$G'(t) = -\frac{q'(t)}{q(t)} G(t),$$

which implies that $t \in \chi$. It follows by (1), (2), (3), (4) that

$$q(t) = p(t)^2 + \tfrac{3}{4} = 1, \tag{6}$$

$$q'(t) = 2p(t)p'(t) = p'(t) = \begin{cases} \lambda(t) & \text{if } \lambda(t) \neq 0, \\ 1 & \text{if } \lambda(t) = 0. \end{cases} \tag{7}$$

On the other hand, since $t \in \chi$, we have

$$G'(t) = \lambda(t)G(t),$$

and, hence, by (5),

$$0 = (q'(t) + q(t)\lambda(t))G(t).$$

Since $r > 0$, we have $G(t) \neq 0$, and it follows that

$$q'(t) + q(t)\lambda(t) = 0.$$

Consequently, by (6) and (7),

$$0 = \lambda(t) + \lambda(t) = 2\lambda(t) \quad \text{if } \lambda(t) \neq 0$$

and

$$0 = 1 + 1 \cdot 0 = 1 \quad \text{if } \lambda(t) = 0.$$

Both cases are impossible. Therefore, $F'(t) \neq 0$ for every $t \in [0, 1]$. □

Theorem 5 (Existence of orthonormal bases of class C^p of the kernel and the image of a rectangular matrix function of q real variables). *Let $\Omega \subseteq \mathbb{R}^q$ be C^p-diffeomorphic to $\mathbb{R}^q$. Let $A \in C^p(\Omega, \mathbb{R}_r^{m \times n})$. Then there exist*

$$u_1, \dots, u_m \in C^p(\Omega, \mathbb{R}^m), \qquad v_1, \dots, v_n \in C^p(\Omega, \mathbb{R}^n)$$

such that, for every $t \in \Omega$,

(a) *if $r > 0$, then $(u_1(t), \dots, u_r(t))$ is an orthonormal basis of* $\operatorname{Im} A(t)$;
(b) *if $r < m$, then $(u_{r+1}(t), \dots, u_m(t))$ is an orthonormal basis of* $(\operatorname{Im} A(t))^\perp$;
(c) *if $r > 0$, then $(v_1(t), \dots, v_r(t))$ is an orthonormal basis of* $(\operatorname{Ker} A(t))^\perp$;
(d) *if $r < n$, then $(v_{r+1}(t), \dots, v_n(t))$ is an orthonormal basis of* $\operatorname{Ker} A(t)$.

Proof. The proof is the same as the proof of Theorem 8.2 of [2] except for the following modifications:

(a) Replace $\mathbb{C}$ by $\mathbb{R}$ in the proof of Theorem 8.2 of [2].
(b) In the proof of Lemma 8.1 of [2], apply Theorem 4 of this paper instead of Theorem 7.2 of [2].
(c) In the proof of Lemma 8.1 of [2], if $\det X(t) < 0$, then multiply the first column of $A(t)$ and $X(t)$ by -1. □

REFERENCES

1. K. Deimling, *Nonlinear functional analysis*, Springer, New York, 1985.
2. J.-Cl. Evard, *On the existence of bases of class C^p of the kernel and the image of a matrix function*, Linear Algebra Appl. **135** (1990), 33–67.
3. J.-Cl. Evard and F. Jafari, *Polynomial path connectedness and Hermite interpolation in topological vector spaces*, submitted.

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