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HÖRMANDER'S CONDITION AND A CONVOLUTION OPERATOR GENERALIZING RIESZ POTENTIALS

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ABSTRACT. Under certain hypotheses including a Hörmander-type condition on the convolution kernel K we show that $K * f$ belongs to the space $\mathrm{BMO}(\mathbf{R}^n)$ whenever f belongs to the space $L^{p,\infty}(\mathbf{R}^n)$ (weak L^p) for certain p.

Let $1 < r < \infty$ and $1/r + 1/r' = 1$. Consider a convolution operator $Tf = K * f$ on $\mathbf{R}^n$ which satisfies the following two conditions:

$$\text{(i)} \qquad \|Tf\|_{q_0} \le B_1 \|f\|_{p_0}$$

for some p_0, q_0 with $1/p_0 = 1/q_0 + 1/r'$, $1 < p_0 < r'$, and

$$\text{(ii)} \qquad \int_{|x|>2|y|} |K(x-y) - K(x)|^r \, dx \le B_2 \quad \forall y \neq 0.$$

Hörmander [H] proved that conditions (i) and (ii) imply that T is of weak type $(1, r)$; hence, by interpolation and duality T is of strong type (p, q) for all p, q with $1/p = 1/q + 1/r'$, $1 < p < r'$. (We wish to thank Mr. Jung Soo Rhee for bringing this result to our attention.) A condition like (ii) is often referred to as Hörmander's condition.

Our present observation is that (i) and (ii) also imply the boundedness of T from $L^{r'}$ to $\mathrm{BMO}(\mathbf{R}^n)$. If (ii) is replaced by a stronger condition (ii′) (see below), we obtain the stronger conclusion that T is bounded from weak $L^{r'}$ to $\mathrm{BMO}(\mathbf{R}^n)$. We state and prove our result in the second form. Our proof relies heavily on the method of proof of the boundedness from L^∞ to BMO of the Calderón-Zygmund singular integral operator (see [S1, GR]). The symbol $\|\cdot\|_{L^{p,q}}$ (or $\|\cdot\|_{p,q}$) denotes the Lorentz space norm (see [O, SW]), and $\|\cdot\|_*$ denotes the $\mathrm{BMO}(\mathbf{R}^n)$ norm (see [GR]).

Theorem. *Suppose that* T *satisfies* (i) (*or merely that* T *is of restricted weak type* (p_0, q_0): (i′) $\|Tf\|_{q_0,\infty} \le B_1 \|f\|_{p_0,1}$) *for some* $1 < r < \infty$. *If* K *satisfies*

$$\text{(ii}'\text{)} \qquad \|K(x-y) - K(x)\|_{L^{r,1}(E_y;\, dx)} \le B_2 \quad \forall y \neq 0,$$

where $E_y = \{x \in \mathbf{R}^n : |x| > 2|y|\}$, *then there exists a constant* A *such that*

$$\|Tf\|_* \le A \|f\|_{L^{r',\infty}(\mathbf{R}^n)}.$$

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Proof. It is enough to show that there exists a constant A such that for every f and cube Q there is a constant a_Q such that

$$\frac{1}{|Q|}\int_Q |Tf(y) - a_Q|\,dy \le \frac{A}{2}\|f\|_{L^{r',\infty}(\mathbf{R}^n)}. \tag{1}$$

Let Q^* denote the cube with the same center as Q, but $2\sqrt{n}$ times the diameter. By translation invariance we may assume that Q and Q^* are centered at the origin. Let $a_Q = \int_{(Q^*)^c} K(-t)f(t)\,dt$. Write $f_1 = f\cdot\chi_{Q^*}$ and $f_2 = f - f_1$. Then

$$Tf(y) - a_Q = Tf_1(y) + \int_{(Q^*)^c}(K(y-t) - K(-t))f(t)\,dt = Tf_1(y) + J.$$

By Hölder's inequality for Lorentz spaces (see [O]) and (i) (or (i$'$)),

$$\begin{aligned}\frac{1}{|Q|}\int_Q |Tf_1(y)|dy &\le |Q|^{-1}\|Tf_1\|_{L^{q_0,\infty}(Q)}\|1\|_{L^{q_0',1}(Q)}\\ &= C|Q|^{-1/q_0}\|Tf_1\|_{L^{q_0,\infty}(Q)} \le CB_1|Q|^{-1/q_0}\|f_1\|_{p_0,1}\\ &\le CB_1|Q|^{-1/q_0}\|\chi_{Q^*}\|_{q_0,1}\|f\|_{r',\infty} \le CB_1\|f\|_{r',\infty}.\end{aligned}$$

Also, by Hölder's inequality and (ii$'$),

$$\begin{aligned}\frac{1}{|Q|}\int_Q |J|\,dy &\le \frac{1}{|Q|}\int_{y\in Q}\int_{|t|>2|y|}|K(y-t) - K(-t)||f(t)|\,dt\,dy\\ &\le \frac{1}{|Q|}\int_Q \|K(y-t) - K(-t)\|_{L^{r,1}(E_y;dt)}\|f\|_{r',\infty}\,dy \le B_2\|f\|_{r',\infty}.\end{aligned}$$

Combining these estimates gives (1) with $A = 2(CB_1 + B_2)$. □

Examples. If $K * f$ is the Riesz potential $I_\alpha(f)$ of order α, $0 < \alpha < n$, then (i) holds with $r' = n/\alpha$ (see [S2]). An easy calculation shows that condition (ii$'$) follows from the estimate $|K(x-y) - K(x)| \le C|y||x|^{\alpha-n-1}$, $|x| > 2|y|$. So it follows that $\|I_\alpha(f)\|_* \le A\|f\|_{L^{n/\alpha,\infty}(\mathbf{R}^n)}$. (The weaker estimate $\|I_\alpha(f)\|_* \le A\|f\|_{L^{n/\alpha}(\mathbf{R}^n)}$ is contained in [A].) An immediate corollary of this result is the estimate $\|J_\alpha(f)\|_* \le A\|f\|_{L^{n/\alpha,\infty}(\mathbf{R}^n)}$ for the Bessel potential $J_\alpha(f)$, which was originally proved by Stein and Zygmund [SZ] (see also [S2, p. 164]).

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