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ORTHOCOMPACTNESS IN INFINITE PRODUCT SPACES

NOBUYUKI KEMOTO AND YUKINOBU YAJIMA

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Dedicated to Professor Akihiro Okuyama on his 60th birthday

ABSTRACT. In this paper, we prove the following results for an infinite product space $X = \prod_{\alpha\in\kappa} X_\alpha$.

(1) If a dense subspace of X is orthocompact, then it is κ-metacompact.

(2) Assume that all finite subproducts of X are hereditarily orthocompact. If a subspace of X is κ-metacompact, then it is orthocompact.

1. INTRODUCTION

Throughout this paper, all spaces are assumed to be regular T_1, κ denotes an infinite cardinal, and all product spaces are infinite product spaces. Whenever we consider a product space $\prod_{\alpha\in\kappa} X_\alpha$, we always assume that each X_α contains at least two points.

Bešlagić [Be] proved that a product space $X = \prod_{\alpha\in\kappa} X_\alpha$ is κ-paracompact if it is normal. Conversely, Aoki [Ao] proved that a product space X is normal (orthocompact) if it is κ-paracompact and each finite subproduct of X is normal (orthocompact). In these connections, we recall Scott's result [S1] that a space Z is κ-metacompact if $Z \times 2^\kappa$ is orthocompact.

In this paper, we prove that a dense subspace of a product space $X = \prod_{\alpha\in\kappa} X_\alpha$ is κ-metacompact if it is orthocompact. Conversely, we also prove that a product space X is orthocompact if it is κ-metacompact and each finite subproduct of X is hereditarily orthocompact. Moreover, we can give various applications of these results.

In the rest of this section, we state notation and basic facts. For a set S and a cardinal λ, we define $[S]^{<\lambda} = \{T \subset S : |T| < \lambda\}$, $[S]^{\leq\lambda} = \{T \subset S : |T| \leq \lambda\}$, and $[S]^\lambda = \{T \subset S : |T| = \lambda\}$, where $|T|$ denotes the cardinality of T. Let $\mathscr{U}$ be a collection of subsets of S and $x \in S$. Then $(\mathscr{U})_x$ denotes $\{U \in \mathscr{U} : x \in U\}$. We say that a collection $\mathscr{V}$ of subsets of S is a *weak refinement* of $\mathscr{U}$ if each member of $\mathscr{V}$ is contained in some member of $\mathscr{U}$. Furthermore, such a $\mathscr{V}$ is a *refinement* of $\mathscr{U}$ if $\bigcup\mathscr{V} = \bigcup\mathscr{U}$.

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Let $X = \prod_{\alpha\in\kappa} X_\alpha$ be a product space. For an $F \subset \kappa$, we denote by $X(F)$ the subproduct $\prod_{\alpha\in F} X_\alpha$ and denote by π_F the canonical projection map $X \to X(F)$. Such an $X(F)$ is called a *finite subproduct* of X if F is finite. In particular, we write X_α and π_α for $X(\{\alpha\})$ and $\pi_{\{\alpha\}}$, respectively.

A space X is (κ-) *metacompact* if each open cover of X (with cardinality $\le \kappa$) has a point-finite open refinement. A space X is (*weakly*) *submetacompact* (or (*weakly*) *θ-refinable*) if for each open cover $\mathscr{U}$ of X there is a sequence $\{\mathscr{V}_n : n \in \omega\}$ of (weak) open refinements of $\mathscr{U}$ such that for each $x \in X$ there is an $n \in \omega$ such that ($x \in \bigcup \mathscr{V}_n$ and) $\mathscr{V}_n$ is point-finite at x. We define (*weak*) *κ-submetacompactness* analogously. In particular, a sequence $\{\mathscr{V}_n : n \in \omega\}$ of covers of X is called a *θ-sequence* if for each $x \in X$ there is an $n \in \omega$ such that $\mathscr{V}_n$ is point-finite at X.

An open cover $\mathscr{V}$ of a space X is *interior-preserving* if $\bigcap \mathscr{V}'$ is open in X for each $\mathscr{V}' \subset \mathscr{V}$. A space X is (κ-) *orthocompact* if every open cover of X (with cardinality $\le \kappa$) has an interior-preserving open refinement. Note that a space X is orthocompact if and only if every open cover $\mathscr{U}$ has an open refinement $\mathscr{V}$ of $\mathscr{U}$ such that $\bigcap(\mathscr{V})_x$ is a (an open) neighborhood of x. A space X is (*weakly*) *suborthocompact* [KY, Ya] if for each open cover $\mathscr{U}$ of X there is a sequence. $\{\mathscr{V}_n : n \in \omega\}$ of (weak) open refinements of $\mathscr{U}$ such that for each $x \in X$ there is an $n \in \omega$ such that $\bigcap(\mathscr{V}_n)_x$ is a neighborhood of x. We define (*weak*) *κ-suborthocompactness* analogously. In particular, a sequence $\{\mathscr{V}_n : n \in \omega\}$ of covers of X is called an *ι-sequence* [KY] if for each $x \in X$ there is an $n \in \omega$ such that $\bigcap(\mathscr{V}_n)_x$ is a neighborhood of x. Clearly, each θ-sequence of open covers of X is an ι-sequence.

By these definitions, the following diagram is easily verified. But note that the ordinal space ω_1 is (hereditarily) orthocompact but not weakly (ω_1-) submetacompact.

$$\begin{array}{ccccc}
\text{metacompact} & \longrightarrow & \text{submetacompact} & \longrightarrow & \text{weakly submetacompact} \\
\downarrow & & \downarrow & & \downarrow \\
\text{orthocompact} & \longrightarrow & \text{suborthocompact} & \longrightarrow & \text{weakly suborthocompact}
\end{array}$$

2. κ-ORTHOCOMPACTNESS IN PRODUCT SPACES

Theorem 2.1. *Let Y be a dense subspace of a product space $X = \prod_{\alpha\in\kappa} X_\alpha$. Then Y is κ-metacompact (κ-submetacompact, weakly κ-submetacompact) if and only if it is κ-orthocompact (κ-suborthocompact, weakly κ-suborthocompact).*

Proof. We prove the "if" part. Let $\mathscr{U} = \{U_\alpha : \alpha \in \kappa\}$ be an open cover of Y with cardinality $\le \kappa$. First we show that $\mathscr{U}$ has an open refinement (in Y) $\mathscr{W}$ of cardinality $\le \kappa$ such that $\operatorname{int}_Y(\bigcap \mathscr{W}') = 0$ for each $\mathscr{W}' \in [\mathscr{W}]^\omega$.

For each $\alpha \in \kappa$, pick distinct two points $p_\alpha(0)$ and $p_\alpha(1)$ in X_α. Since X_α is regular T_1, we take an open neighborhood $N_\alpha(i)$ of $p_\alpha(i)$, where $i \in 2 = \{0, 1\}$, such that $X_\alpha = N_\alpha(0) \cup N_\alpha(1)$ and $p_\alpha(1-i) \notin \operatorname{cl}_{X_\alpha} N_\alpha(i)$ for each $\alpha \in \kappa$ and each $i \in 2$. Let $G_\alpha(i) = \pi_\alpha^{-1}(N_\alpha(i)) \cap Y$ for each $\alpha \in \kappa$ and each $i \in 2$. Note that each $G_\alpha(i)$ is open in Y and $Y = G_\alpha(0) \cup G_\alpha(1)$ for each $\alpha \in \kappa$.

Claim. $\operatorname{int}_Y(\bigcap_{\alpha\in A} G_\alpha(i)) = 0$ for each $A \in [\kappa]^\omega$ and each $i \in 2$.

Proof. Assume that there are an $A \in [\kappa]^{\omega}$, an $i \in 2$, and a $y \in Y$ such that $y \in \mathrm{int}_Y(\bigcap_{\alpha\in A} G_\alpha(i))$. Then there are an $F \in [\kappa]^{<\omega}$ and an open set V in $X(F)$ such that $y \in \pi_F^{-1}(V) \cap Y \subset \bigcap_{\alpha\in A} G_\alpha(i)$. Since F is finite and A is infinite, pick a β in $A - F$. Since Y is dense in X and $\beta \notin F$, there is a point z in $\pi_F^{-1}(V) \cap \pi_\beta^{-1}(X_\beta - \mathrm{cl}\, N_\beta(i)) \cap Y$. Then we have $z \in \pi_F^{-1}(V) \cap Y \subset G_\beta(i) \subset \pi_\beta^{-1}(N_\beta(i))$. This contradicts $z \in \pi_\beta^{-1}(X_\beta - \mathrm{cl}\, N_\beta(i))$ and completes the proof of the claim.

Put $\mathscr{W} = \{U_\alpha \cap G_\alpha(i) : \alpha \in \kappa \text{ and } i \in 2\}$. Then it follows from the claim that $\mathscr{W}$ is a desired open refinement of $\mathscr{U}$.

Now, we prove only the second case. From the above argument, we may assume that $\mathscr{U} = \{U_\alpha : \alpha \in \kappa\}$ is an open cover of Y such that $\mathrm{int}_Y(\bigcap \mathscr{U}') = 0$ for each $\mathscr{U}' \in [\mathscr{U}]^{\omega}$. There is an ι-sequence $\{\mathscr{V}_n : n \in \omega\}$ of open refinements of $\mathscr{U}$. Then it is easy to see that each $\mathscr{V}_n$ may be assumed to be a precise open refinement of $\mathscr{U}$, that is, $\mathscr{V}_n = \{V_n(U) : U \in \mathscr{U}\}$ such that $V_n(U) \subset U$ for each $U \in \mathscr{U}$. Pick an $x \in Y$. Choose an $n \in \omega$ such that $\bigcap(\mathscr{V}_n)_x$ is a neighborhood of x. Assume that $(\mathscr{V}_n)_x$ is infinite. Then there is some $\mathscr{U}' \in [\mathscr{U}]^{\omega}$ such that $\{V_n(U) : U \in \mathscr{U}'\} \subset (\mathscr{V}_n)_x$. So we have

$$x \in \mathrm{int}_Y\left(\bigcap(\mathscr{V}_n)_x\right) \subset \mathrm{int}_Y\left(\bigcap\{V_n(U) : U \in \mathscr{U}'\}\right) \subset \mathrm{int}_Y\left(\bigcap \mathscr{U}'\right) = 0.$$

This is a contradiction. Thus, $\{\mathscr{V}_n : n \in \omega\}$ is a θ-sequence of open refinements of $\mathscr{U}$. The proof is complete.

Corollary 2.2. *If a product space $X = \prod_{\alpha\in\kappa} X_\alpha$ is orthocompact (suborthocompact, weakly suborthocompact), then X is κ-metacompact (κ-submetacompact, weakly κ-submetacompact).*

For a space X, $L(X)$ denotes the Lindelöf degree of X.

Corollary 2.3 [S1, Ya]. *A space X is metacompact (submetacompact, weakly submetacompact) if and only if $X \times 2^{\kappa}$ is orthocompact (suborthocompact, weakly suborthocompact) where $L(X) \le \kappa$.*

Remark. Moreover, we can easily obtain the analogies of [Ao, Theorem 3.1]: A space X is (weakly) κ-submetacompact if and only if $X \times A(\kappa)$ is (weakly) κ-suborthocompact, where $A(\kappa)$ is the one-point compactification of a discrete space of cardinality κ. Observe that this is a generalization of Corollary 2.3.

It is known that ω^{ω_1} is not orthocompact; see [Ao, Theorem 3.4] or [S2, Theorem 2.4]. Moreover, we have

Corollary 2.4. *ω^{ω_1} is not suborthocompact.*

Proof. Assume that $X = \omega^{\omega_1}$ is suborthocompact. Then, by Corollary 2.2, X is ω_1-submetacompact. Since the weight of X is ω_1, it is submetacompact. But it follows from the statement in [PP, p. 63] that X is not submetacompact. This is a contradiction.

Let Y be a Σ-product of $\{X_\alpha : \alpha \in \kappa\}$. Then Y is said to be *proper* [Pr, §7] if Y is a proper subspace of $\prod_{\alpha\in\kappa} X_\alpha$ (i.e., $\kappa \ge \omega_1$ and $|X_\alpha| \ge 2$ for each $\alpha \in \kappa$).

Corollary 2.5. *All proper Σ-products are not weakly suborthocompact.*

Proof. Let Y be a proper Σ-product of $\{X_\alpha : \alpha \in \kappa\}$, where $\kappa \ge \omega_1$. Assume that Y is weakly suborthocompact. Since Y is dense in X, it follows from

Theorem 2.1 that Y is weakly κ-submetacompact. Since Y contains a closed subspace which is homeomorphic to the ordinal space ω_1 (cf. [Pr, Proposition 7.2]), the space ω_1 is weakly κ-submetacompact. But it is well known that the space ω_1 is not weakly ω_1-submetacompact. This is a contradiction.

Corollary 2.6. *Let X be a product space of paracompact p-spaces (e.g., metrizable spaces). Then the following are equivalent.*

(1) *X is (sub)orthocompact.*
(2) *X is normal.*
(3) *X is paracompact.*

Using Corollary 2.4, the proof is similar to that of [Pr, Corollary 6.5].

Remark. The condition "paracompact p-space" in Corollary 2.6 is essential. In fact, let X be a Σ-product in 2^{ω_1}. Then X is homeomorphic to X^ω. It follows from [Pr, Theorem 7.4] and Corollary 2.5 that X^ω is normal but not weakly suborthocompact.

We obtain the following generalization of [Ao, Theorem 3.5] or [S2, Theorem 2.5].

Corollary 2.7. *The following are equivalent for a space X.*

(1) *X is compact.*
(2) *X^κ is suborthocompact for any cardinal κ.*
(3) *X^κ is suborthocompact for some cardinal κ with $\kappa \geq \omega_1 \cdot L(X)$.*

Using Corollaries 2.3 and 2.4, the proof is parallel to that of [Ao, Theorem 3.5].

If ω^{ω_1} was not weakly submetacompact, then "suborthocompact" in most of our corollaries could be replaced by "weakly suborthocompact". Hence, we conclude this section with the following problem.

Problem 2.8. Is ω^{ω_1} not weakly submetacompact?

3. κ-METACOMPACTNESS IN PRODUCT SPACES

As the converse of Corollary 2.2, we obtain the following:

Theorem 3.1. *Assume that all finite subproducts of a product space $X = \prod_{\alpha\in\kappa} X_\alpha$ are hereditarily orthocompact. If a subspace Y of X is κ-metacompact (κ-submetacompact, weakly κ-submetacompact), then it is orthocompact (suborthocompact, weakly suborthocompact).*

Proof. We prove only the second case. Let $\mathscr{U}$ be an open cover of Y. We may assume that, for each $U \in \mathscr{U}$, there are an $F(U) \in [\kappa]^{<\omega}$ and an open set $G(U)$ in $X(F(U))$ such that $U = \pi_{F(U)}^{-1}(G(U)) \cap Y$. For each $F \in [\kappa]^{<\omega}$, put $\mathscr{U}_F = \{U \in \mathscr{U} : F(U) = F\}$ and $G_F = \bigcup\{G(U) : U \in \mathscr{U}_F\}$. Then it is easy to check that each G_F is open in $X(F)$ and $\mathscr{A} = \{\pi_F^{-1}(G_F) \cap Y : F \in [\kappa]^{<\omega}\}$ is an open cover of Y. Since each $X(F)$ is hereditarily orthocompact, there is an interior-preserving collection $\mathscr{B}(F) = \{B_F(U) : U \in \mathscr{U}_F\}$ of open sets in $X(F)$ such that $B_F(U) \subset G(U)$ for each $U \in \mathscr{U}_F$ and $\bigcup \mathscr{B}(F) = G_F$. By the κ-submetacompactness of Y and $|\mathscr{A}| \leq \kappa$, there is a θ-sequence $\{\mathscr{V}_n : n \in \omega\}$ of open refinements of $\mathscr{A}$. We may assume that $\mathscr{V}_n = \{V_F^n : F \in [\kappa]^{<\omega}\}$

such that $V_F^n \subset \pi_F^{-1}(G_F) \cap Y$ for each $F \in [\kappa]^{<\omega}$ and each $n \in \omega$. Put $\mathscr{W}_F^n = \{\pi_F^{-1}(B_F(U)) \cap V_F^n : U \in \mathscr{U}_F\}$ for each $F \in [\kappa]^{<\omega}$ and each $n \in \omega$. Then it is easy to check that each $\mathscr{W}_F^n$ is an interior-preserving collection of open sets in Y whose union is V_F^n. Put $\mathscr{W}_n = \bigcup\{\mathscr{W}_F^n : F \in [\kappa]^{<\omega}\}$ for each $n \in \omega$. Observe that each $\mathscr{W}_n$ is an open refinement of $\mathscr{U}$. We show that $\{\mathscr{W}_n : n \in \omega\}$ is an ι-sequence. Pick an $x \in X$. Since $\{\mathscr{V}_n : n \in \omega\}$ is a θ-sequence, take an $n \in \omega$ such that $(\mathscr{V}_n)_x$ is finite, say $(\mathscr{V}_n)_x = \{V_F^n : F \in \mathscr{F}\}$ for some $\mathscr{F} \in [[\kappa]^{<\omega}]^{<\omega}$. Since $\mathscr{W}_F^n$ is interior-preserving and $x \in V_F^n = \bigcup \mathscr{W}_F^n$, $\bigcap(\mathscr{W}_F^n)_x$ is an open neighborhood of x for each $F \in \mathscr{F}$. Since $(\mathscr{W}_n)_x = \bigcup_{F \in \mathscr{F}}(\mathscr{W}_F^n)_x$ and $|\mathscr{F}| < \omega$, it follows that $\bigcap(\mathscr{W}_n)_x = \bigcap_{F \in \mathscr{F}}(\bigcap(\mathscr{W}_F^n)_x)$ is an open neighborhood of x. This completes the proof.

Considering [Ao, Corollary 2.5], it is natural to raise

Problem 3.2. If a product space $X = \prod_{\alpha \in \kappa} X_\alpha$ is κ-metacompact and all finite subproducts of X are orthocompact, is X orthocompact?

Proposition 3.3. *Assume that all finite subproducts of a product space* $X = \prod_{\alpha \in \kappa} X_\alpha$ *are hereditarily metacompact. If a dense subspace* Y *of* X *is* κ-*orthocompact* (κ-*suborthocompact, weakly* κ-*suborthocompact*), *then* Y *is metacompact* (*submetacompact, weakly submetacompact*).

Proof. The second case: Observe that Y is κ-submetacompact according to Theorem 2.1, because Y is a κ-suborthocompact dense subspace of X. Then replacing "interior-preserving" by "point-finite" in the proof of Theorem 3.1, we can prove similarly.

Corollary 3.4. *Let* X *be a product space of metrizable spaces and* Y *a dense subspace of* X. *Then* Y *is orthocompact* (*suborthocompact, weakly suborthocompact*) *if and only if it is metacompact* (*submetacompact, weakly submetacompact*).

We can consider this is an analogue of [Ba, Theorem 1].

Remark. Under the assumption of Corollary 3.4, X is normal if and only if it is paracompact (see Corollary 2.6). But one cannot replace "orthocompact" and "metacompact" by "normal" and "paracompact", respectively, in Corollary 3.4. In fact, let Y be a Σ-product in $X = 2^{\omega_1}$. Then Y is a normal nonparacompact, dense subspace of X.

References

[Ao] Y. Aoki, *Orthocompactness of inverse limits and products*, Tsukuba J. Math. **4** (1980), 241–255.

[Ba] D. P. Baturov, *Normality in dense subspaces of products*, Topology Appl. **36** (1990), 111–116.

[Be] A. Bešlagić, *Normality in products*, Topology Appl. **22** (1986), 71–82.

[KY] N. Kemoto and Y. Yajima, *Orthocompactness in products*, Tsukuba J. Math. **16** (1992), 407–422.

[PP] R. Pol and E. Puzio-Pol, *Remarks on Cartesian products*, Fund. Math. **93** (1976), 57–69.

[Pr] T. C. Przymusiński, *Products of normal spaces*, Handbook of Set Theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North-Holland, Amsterdam, 1984, pp. 781–826.

[S1] B. M. Scott, *Toward a product theory for orthocompactness*, Studies in Topology (N. M. Stavrakas and K. R. Allen, eds.), Academic Press, New York, 1975, pp. 517–537.

[S2] ——, *More about orthocompactness*, Topology Proc. **5** (1980), 155–184.

[Ya] Y. Yajima, *A characterization of submetacompactness in terms of products*, Proc. Amer. Math. Soc. **112** (1991), 291–296.

Department of Mathematics, Faculty of Education, Oita University, Dannoharu, Oita, 870-11 Japan

E-mail address: `nkemoto@cc.oita-u.ac.jp`

Department of Mathematics, Kanagawa University, Yokohama, 221 Japan

E-mail address: `yuki@kani.cc.kanagawa-u.ac.jp`