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AN INVERSE SPECTRAL PROBLEM FOR STURM-LIOUVILLE OPERATORS WITH DISCONTINUOUS COEFFICIENTS

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ABSTRACT. For generic Sturm-Liouville problems with piecewise constant leading coefficients, the leading coefficient can be determined up to a finite ambiguity from the eigenvalues of the problem.

1. INTRODUCTION

Ordinary differential operators with discontinuous leading coefficients occasionally arise in scientific modelling; for instance [11] and [8] discuss such a geophysical application, while [10] gives examples in electromagnetics and elasticity. This paper addresses the inverse spectral problem of recovering information about the coefficients of Sturm-Liouville operators from the sequence of eigenvalues. The case we analyze is the case when the leading coefficient has a finite number of jump discontinuities. Our aim is to determine, to the extent possible, the location and magnitude of these jumps from the spectrum of a single Sturm-Liouville problem.

Inverse eigenvalue problems, usually for equations of the form

$$-y'' + q(x)y = \lambda y$$

with a variety of boundary conditions, have an extensive literature. Unless $q(x)$ is constrained, one needs the spectra from two sets of boundary conditions, or one spectrum and a sequence of norming constants, to uniquely determine $q(x)$. Early important work on these problems was done by Borg [3], followed shortly thereafter by extensive work in the Soviet Union [6, 9]. More recently, new geometric ideas were introduced by McKean and Trubowitz and their coworkers (see [15] and the references therein). There have been several works addressing inverse eigenvalue problems with less-regular coefficients. Most of these are motivated by geophysical models for oscillations of the earth. Hald [8] examines problems with an interior jump condition. Andersson [1] and Coleman and McLaughlin [5] consider problems of the form

$$(p^2y')' + \lambda p^2 y = 0$$

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with coefficients too irregular for the classical reduction to Liouville normal form. For surveys of these and related results and extensive lists of references one can consult [12] and [4].

The problem we consider is

$$\text{(1.a)}\qquad \begin{gathered} -[1/p(x)^2]\varphi'' + q(x)\varphi = \lambda\varphi\,, \\ a\varphi'(0) - b\varphi(0) = 0\,, \qquad c\varphi'(1) - d\varphi(1) = 0 \end{gathered}$$

with $a^2 + b^2$ and $c^2 + d^2$ positive, where $q(x) \in L^1[0, 1]$ and $p(x)$ is a piecewise constant positive function with value p_n on the intervals (x_n, x_{n+1}), $n = 0, \dots, N-1$, and with $0 = x_0 < x_1 < \cdots < x_N = 1$. We require that $\varphi(x)$ and $\varphi'(x)$ are continuous on $[0, 1]$ and satisfy the differential equation on the open intervals (x_n, x_{n+1}).

Elementary examples show that in general p cannot be determined from the eigenvalues of the problem (1.a). If $q(x) = 0$ and $a = c = 0$, then the two problems with coefficients $p_1(x)$ and $p_2(x)$ will have the same eigenvalues if $p_1(x) = p_2(1-x)$. A more interesting family of examples on the interval $[0, \pi]$ was pointed out in [7]. If $0 < \alpha < 1$, it is easy to show that the following problems all have the same eigenvalues:

$$\text{(1.b)}\qquad -[1/p(x)^2]y'' = \lambda y\,, \qquad y(0) = 0 = y(\pi)$$

where $p(x) = 1/(2\alpha)$ for $0 < x < \alpha\pi$ and $p(x) = 1/(2(1-\alpha))$ for $\alpha\pi < x < \pi$.

To state the main result it will be convenient to introduce some notation. Let $l_n = x_{n+1} - x_n$ and $\zeta_n = l_n p_n$, $n = 0, \dots, N-1$. Let $\mu_n = p_n/p_{n+1}$ for $n = 0, \dots, N-2$. Denote by $B_i = (\beta(0, i), \dots, \beta(N-1, i))$ an N-tuple whose entries have values ± 1. One of our hypotheses will be that the numbers $\sum_{m=0}^{N-1} \beta(m, i)\zeta_m$ are distinct if the vectors B_i are distinct. Notice that for distinct B_i two such sums agree when $(\zeta_0, \dots, \zeta_{N-1})$ lies in a hyperplane, so asking for the sums to be distinct is the condition that $(\zeta_0, \dots, \zeta_{N-1})$ lie outside a finite number of hyperplanes.

Theorem 1.1. *Suppose* $(\zeta_0, \dots, \zeta_{N-1})$ *is such that the sums* $\sum_{m=0}^{N-1} \beta(m, i)\zeta_m$ *are distinct if* $(\beta(0, i), \dots, \beta(N-1, i))$ *are distinct vectors with entries* ± 1. *Suppose that* $\mu_n \neq 1$ *for* $n = 0, \dots, N-2$. *If two problems* (1.a) *have the same eigenvalues, then the interval lengths and corresponding constants* (l_n, p_n), $n = 0, \dots, N-1$, *are the same up to a permutation of* $0, \dots, N-1$.

In outline form, the proof of Theorem 1.1 is straightforward. As in the case with $p = 1$ the eigenvalues of problem (1.a) are the zeros of an entire function $F_q(\lambda)$, which has order $\frac{1}{2}$ and only simple zeros (Lemma 3.1); by Hadamard's theorem [16, p. 74] $F_q(\lambda)$ is determined up to a constant multiple by its zeros. Problem (1.a) can be treated as a perturbation of the case when $q = 0$. For the case $q = 0$ it is possible to find an explicit formula for $F_0(\lambda)$. In fact, it has the form

$$\begin{aligned} F_0(\lambda) = \sum_{i \in I} e_i\{&\beta(N-1, i)ac\mu_{N-1}\omega\sin(f_i\omega) \\ &+ [ad + \beta(N-1, i)bc\mu_{N-1}/p_0]\cos(f_i\omega) + bd\sin(f_i\omega)/[p_0\omega]\} \end{aligned}$$

where $f_i = \sum_{m=0}^{N-1} \beta(m, i)\zeta_m$ and $\omega = \lambda^{1/2}$. The frequencies f_i and the coefficients can be determined by considering expressions like

$$\lim_{L\to\infty}[2/L]\int_1^L F_0(\lambda)\sin(\nu\omega)/\omega\, d\omega$$

for $\nu > 0$. Some elementary algebra then gives the result.

By a change of variables it is possible to show that more general boundary value problems will have the same eigenvalues as one of the problems (1.a). Consider first a problem of the form

$$\begin{gathered} -R(x)[P_1(x)u']' + Q_1(x)u = \lambda u, \\ k_1u'(0) + k_2u(0) = 0, \qquad k_3u'(1) + k_4u(1) = 0. \end{gathered} \tag{1.c}$$

It is assumed that $R(x)$, $1/R(x)$, $P_1(x)$, and $1/P_1(x)$ are positive and bounded and that $P_1(x)$ is continuous except for a finite number of jump discontinuities. It is pointed out in [13] that the change of variables $t = [\int^x P^{-1}(s)]/[\int_0^1 P^{-1}(s)\,ds]$ and $\varphi(t) = u(x(t))$ leads to an equation $-P_2(x)\varphi'' + Q_2(x)\varphi = \lambda\varphi$ with a simple change in end point boundary conditions and the location of the discontinuities. Also the jump conditions for u natural at the discontinuities of problem (1.c) become simply the requirement that φ be continuous with continuous first derivative everywhere. To further simplify the problem, note that if $P_2(x)$ is continuous except for finitely many jump discontinuities, then $P_2(x) = p(x)J(x)$ where $J(x)$ is continuous while $p(x)$ is piecewise constant with jumps at the same locations as $P_2(x)$. Finally a standard change of variables [2, p. 296] reduces the problem to the form (1.a).

2. The case $q(x) = 0$

The analysis of problem (1.a) will begin with the case $q(x) = 0$. Let $\omega = \lambda^{1/2}$ with $\omega > 0$ when $\lambda > 0$. On the interval (x_n, x_{n+1}) any solution of $-y'' = \lambda p(x)^2 y$ must have the form

$$y(x) = A_n\cos(p_n\omega[x - x_n]) + (B_n/\omega)\sin(p_n\omega[x - x_n]).$$

There is a linear map from the coefficients

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} A_{n+1} \\ B_{n+1} \end{bmatrix}$$

which can be computed using the continuity of y and y'. The matrix taking the basis elements $\cos(p_n\omega[x-x_n])$ and $\sin(p_n\omega[x-x_n])/\omega$ to a linear combination of $\cos(p_{n+1}\omega[x - x_{n+1}])$ and $\sin(p_{n+1}\omega[x - x_{n+1}])/\omega$ is

$$T_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$$

where

$$\begin{aligned} a_n &= \cos(p_n\omega l_n), & b_n &= \sin(p_n\omega l_n)/\omega, \\ c_n &= -\omega p_n\sin(p_n\omega l_n)/p_{n+1}, & d_n &= p_n\cos(p_n\omega l_n)/p_{n+1}. \end{aligned}$$

Notice that the entries of T_n are entire functions of λ and that

$$T_n = \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix} \begin{bmatrix} \cos(p_nl_n\omega) & \sin(p_nl_n\omega) \\ -p_n\sin(p_nl_n\omega)/p_{n+1} & p_n\cos(p_nl_n\omega)/p_{n+1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/\omega \end{bmatrix}.$$

Let $G(\omega) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \omega \end{smallmatrix}\right]$ and $G^{-1}(\omega) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1/\omega \end{smallmatrix}\right]$.

The following lemma is the result of direct computation.

Lemma 2.1.

$$\begin{bmatrix} \cos(b) & \sin(b) \\ -\mu_b \sin(b) & \mu_b \cos(b) \end{bmatrix} \begin{bmatrix} \cos(a) & \sin(a) \\ \mp\mu_a \sin(a) & \pm\mu_a \cos(a) \end{bmatrix}$$
$$= \frac{(1 \pm \mu_a)}{2} \begin{bmatrix} \cos(a+b) & \sin(a+b) \\ -\mu_b \sin(a+b) & \mu_b \cos(a+b) \end{bmatrix}$$
$$+ \frac{(1 \mp \mu_a)}{2} \begin{bmatrix} \cos(a-b) & \sin(a-b) \\ \mu_b \sin(a-b) & -\mu_b \cos(a-b) \end{bmatrix}.$$

Let

$$M(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}.$$

The next lemma provides a representation for the composition $T_n \cdots T_0$.

Lemma 2.2. *For* $n > 0$, $T_n \cdots T_0 = G(\omega)[\sum_{i \in I_n} C_i] G^{-1}(\omega)$ *where the index set* I_n *is the set of distinct vectors* $B_i = (\beta(0, i), \beta(1, i), \dots, \beta(n, i))$ *with* $\beta(0, i) = 1$, $\beta(m, i) = \pm 1$ *for* $m = 1, \dots, n$ *and the matrices* C_i *are given by*

$$C_i = 2^{-n} \left[\prod_{m=0}^{n-1} (1 + \beta(m, i)\beta(m+1, i)\mu_m) \right] \tag{2.a}$$
$$\times \begin{bmatrix} 1 & 0 \\ 0 & \beta(n, i)\mu_n \end{bmatrix} M\left(\omega \left[\sum_{m=0}^{n} \beta(m, i)\zeta_m\right]\right).$$

Proof. The proof is by induction, using Lemma 2.1 as the essential tool. The initial case $n = 1$ is easily checked. Let $S_n = G^{-1}(\omega) T_n G(\omega)$. When we multiply $T_{n-1} \cdots T_0$ on the left by T_n, we find

$$T_n T_{n-1} \cdots T_0 = G(\omega) \left[\sum_{i \in I_{n-1}} S_n C_i \right] G^{-1}(\omega),$$

with I_{n-1} running over the distinct vectors $(\beta(0, i), \dots, \beta(n-1, i))$. Putting aside for the moment the scalar factor $2^{-n+1} \prod_{m=0}^{n-2} (1 + \beta(m, i)\beta(m+1, i))$, we examine the product

$$S_n \begin{bmatrix} 1 & 0 \\ 0 & \beta(n-1, i)\mu_{n-1} \end{bmatrix} M\left(\omega \left[\sum_{m=0}^{n-1} \beta(m, i)\zeta_m\right]\right),$$

which by Lemma 2.1 is equal to
(2.b)

$$[(1 + \beta(n-1, i)\mu_{n-1})/2] \begin{bmatrix} 1 & 0 \\ 0 & \mu_n \end{bmatrix} M\left(\omega \left[\zeta_n + \sum_{m=0}^{n-1} \beta(m, i)\zeta_m\right]\right)$$
$$+ [(1 - \beta(n-1, i)\mu_{n-1})/2] \begin{bmatrix} 1 & 0 \\ 0 & -\mu_n \end{bmatrix} M\left(\omega \left[-\zeta_n + \sum_{m=0}^{n-1} \beta(m, i)\zeta_m\right]\right).$$

Examining (2.a) we see that the power of 2 is correct. Now each of the vectors $(\beta(0, i), \dots, \beta(n, i))$ appearing in (2.a) is just one of the vectors $(\beta(0, i), \dots, \beta(n-1, i))$ augmented in the last place with ± 1. It is exactly this augmentation that we see in (2.b), establishing the lemma. $\square$

Next the eigenvalues of the $q(x) = 0$ problem will be expressed as the zeros of an elementary function. Using the boundary conditions $ay'(0) - by(0) = 0$ and $cy'(0) - dy(1) = 0$, we see that for all λ any eigenfunction has the form

$$\begin{bmatrix} a \\ b/p_0 \end{bmatrix}$$

in the basis $\cos(p_0\omega x)$, $\sin(p_0\omega x)/\omega$ which is valid in the first interval. On the last interval the eigenfunction is represented by the vector

$$T_{N-2}\cdots T_0 \begin{bmatrix} a \\ b/p_0 \end{bmatrix}.$$

Writing $T_{N-2}\cdots T_0$ as

$$\tau(\lambda) = \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix}$$

we get our eigenfunction

$$\begin{aligned}[\tau_{11}a + \tau_{12}b/p_0]&\cos(p_{N-1}\omega[x - x_{N-1}]) \\ &+ [\tau_{21}a + \tau_{22}b/p_0]\sin(p_{N-1}\omega[x - x_{N-1}])/\omega\end{aligned}$$

and the equation defining the eigenvalues is $F_0(\lambda) = 0$ where

$$\begin{aligned}F_0(\lambda) = {} & d[\tau_{11}a + \tau_{12}b/p_0]\cos(p_{N-1}\omega l_{N-1}]) \\ &+ d[\tau_{21}a + \tau_{22}b/p_0]\sin(p_{N-1}\omega l_{N-1}])/\omega \\ &+ c[\tau_{11}a + \tau_{12}b/p_0]p_{N-1}\omega\sin(p_{N-1}\omega l_{N-1}]) \\ &- c[\tau_{21}a + \tau_{22}b/p_0]p_{N-1}\cos(p_{N-1}\omega l_{N-1}]).\end{aligned}$$

If we define $p_N = 1$ and $\mu_{N-1} = p_{N-1}$ then it is a simple computation to verify that

$$F_0(\lambda) = [d\,,\, -c]T_{N-1}\cdots T_0 \begin{bmatrix} a \\ b/p_0 \end{bmatrix}.$$

It is also elementary that $F_0(\lambda)$ is an entire function of order $1/2$.

It will also be helpful to have a representation for the eigenfunctions $y_1(x\,,\lambda)$, $y_2(x\,,\lambda)$ which satisfy the equation $-(1/p^2)D^2y = \lambda y$ and the usual initial conditions $y_j^{(i-1)}(0\,,\lambda) = \delta_{ij}$, $i\,,j = 1\,,2$. For any $x \in [0\,,1]$, x will lie in an interval $[x_n\,,x_{n+1}]$. In this interval the eigenfunctions will have the form

$$y_1(x\,,\lambda) = [\cos(p_n\omega[x - x_n])\,,\,\sin(p_n\omega[x - x_n])/\omega]T_{n-1}\cdots T_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$y_2(x\,,\lambda) = [\cos(p_n\omega[x - x_n])\,,\,\sin(p_n\omega[x - x_n])/\omega]T_{n-1}\cdots T_0 \begin{bmatrix} 0 \\ 1/p_0 \end{bmatrix}.$$

Using the expression for $T_{n-1}\cdots T_0$ from Lemma 2.2 it is easy to compute that

$$\begin{aligned}y_1(x\,,\lambda) = {} & \sum[c_k\cos(\omega f_k)\cos(p_n\omega[x - x_n]) \\ &\qquad + C_k\omega\sin(\omega f_k)\sin(p_n\omega[x - x_n])/\omega], \\ y_2(x\,,\lambda) = {} & \sum[d_k\cos(\omega f_k)\sin(p_n\omega[x - x_n])/\omega \\ &\qquad + D_k\sin(\omega f_k)\cos(p_n\omega[x - x_n])/\omega]\end{aligned} \tag{2.c}$$

where the terms c_k, C_k, d_k, D_k, and f_k depend on the interval $[x_n\,,x_{n+1}]$ and on the values μ_m and ζ_m but not on λ.

Theorem 1.1 will now be proven in the special case $q(x) = 0$. Writing

$$F_0(\lambda) = [d\,,\,-c]G(\omega)\left[\sum_{i\in I_{N-1}} C_i\right] G^{-1}(\omega)\begin{bmatrix} a \\ b/p_0 \end{bmatrix}$$

and letting $e_i = 2^{-N+1}\prod_{m=0}^{N-2}(1+\beta(m\,,\,i)\beta(m+1\,,\,i)\mu_m)$ for $i \in I_{N-1}$ we find that

$$\begin{aligned} F_0(\lambda) = \sum_{i\in I_{N-1}} e_i\{&\beta(N-1\,,\,i)ac\mu_{N-1}\omega\sin(f_i\omega) \\ &+ [ad+\beta(N-1\,,\,i)bc\mu_{N-1}/p_0]\cos(f_i\omega) \\ &+ bd\sin(f_i\omega)/[p_0\omega]\} \end{aligned}$$

where $f_i = \sum_{m=0}^{N-1}\beta(m\,,\,i)\zeta_m$. Notice, in particular, that $e_i \neq 0$ for all $i \in I_{N-1}$ and that if $a^2+b^2>0$ and $c^2+d^2>0$ then at least one of $ac\mu_{N-1}$, $[ad+\beta(N-1\,,\,i)bc\mu_{N-1}/p_0]$, or bd/p_0 is not zero. In short, all the frequencies which appear to be present are in fact there.

There is an algorithm for recovering, up to a permutation, the function $p(x)$ from the eigenvalues of problem (1.a). Consider

$$\lim_{L\to\infty}[2/L]\int_1^L F_0(\lambda)\omega^{-1}\sin(\nu\omega)\,d\omega \tag{2.d}$$

for ν positive. These limits will all be zero if and only if $ac = 0$. If $ac \neq 0$, the frequencies f_i are determined, and the coefficients e_i are determined up to a single scalar multiple. If the limits in (2.d) are all zero then this data is recovered from

$$\lim_{L\to\infty}[2/L]\int_1^L F_0(\lambda)\cos(\nu\omega)\,d\omega$$

or from

$$\lim_{L\to\infty}[2/L]\int_1^L F_0(\lambda)\omega\sin(\nu\omega)\,d\omega.$$

Because of the assumption that the frequencies f_i are distinct, it is easy to determine the numbers ζ_m, up to a permutation, from the f_i. For convenience of notation assume that $\zeta_0 < \cdots < \zeta_{N-1}$. Let $S = \sum\zeta_m$ and $S_k = [\sum\zeta_m] - 2\zeta_k$. Since the ζ_m are all positive, the largest two observed positive frequencies are S and S_0; their difference is $2\zeta_0$. Proceeding inductively, suppose we have identified $\zeta_0\,,\,\dots\,,\,\zeta_k$. Discard all observed positive frequencies of the form $\sum_{m=k+1}^{N-1}\zeta_m + \sum_{m=0}^{k}(\pm)\zeta_m$. The largest remaining frequency must be S_{k+1}, which gives us ζ_{k+1}. Of course, this procedure only determines $\zeta_0\,,\,\dots\,,\,\zeta_{N-1}$ up to a permutation since their size order is unknown.

The remaining steps will be carried out for each permutation of $\zeta_0,\dots,\zeta_{N-1}$. With each hypothesized permutation we can generate the distinct frequencies $\sum\beta(m\,,\,i)\zeta_m$ and find the (hypothesized) coefficients e_i, $i \in I_{N-1}$. For each m, $0 \le m \le N-1$, examine the two coefficients ε_k obtained when i is selected so that $\beta(0\,,\,1) = \cdots = \beta(k\,,\,i) = 1$, $\beta(k+1\,,\,i) = \cdots = \beta(N-1\,,\,i) = -1$, and e obtained when $\beta(0\,,\,i) = \cdots = \beta(N-1\,,\,i) = 1$. Notice that ε_k will have factors $(1+\mu_m)$ except for the single factor $(1-\mu_k)$ and that the corresponding factors of e will all be $(1+\mu_m)$. Defining $E_k = \varepsilon_k/e$, we find $[1-\mu_k]/[1+\mu_k] = E_k$ or $\mu_k = [1-E_k]/[1+E_k]$. Thus for each of our candidate permutations we can identify $\mu_0, \dots, \mu_{N-2}$. (Note that the unknown scalar dropped out.)

So far we have determined $\zeta_0, \dots, \zeta_{N-1}$ up to permutation and, for each permutation, the hypothesized values of $\mu_0, \dots, \mu_{N-2}$. To finish note that the total length of the interval is $1 = \sum_{m=0}^{N-1} l_m$. We have $p_m = p_0/[\mu_0 \cdots \mu_{m-1}]$ and $\zeta_m = p_m l_m$, so that $1 = \sum l_m = \sum \zeta_m/p_m = [1/p_0] \sum_{m=0}^{N-1} \zeta_m \{\prod_{j=0}^{m-1} \mu_j\}$ gives us p_0. Finally, for each hypothesized permutation the corresponding operator (with $q(x) = 0$) can be constructed, and those giving the correct set of eigenvalues (or $F(\lambda)$) determined.

3. The general case

Let $\varphi(x, \lambda)$ be the solution of the initial value problem

$$-[1/p^2(x)]\varphi'' + q(x)\varphi = \lambda\varphi, \qquad \varphi(0, \lambda) = a, \qquad \varphi'(0, \lambda) = b, \tag{3.a}$$

and let $F_q(\lambda) = d\varphi(1, \lambda) - c\varphi'(1, \lambda)$. $F_q(1, \lambda)$ is an entire function whose zeros are the eigenvalues of problem (1.a). Define $z(x, \lambda) = \partial\varphi(x, \lambda)/\partial\lambda$, and note that z satisfies the initial value problem

$$-[1/p^2(x)]z'' + q(x)z = \varphi + \lambda z, \qquad z(0, \lambda) = 0, \qquad z'(0, \lambda) = 0. \tag{3.b}$$

Following part of the proof of [15, Theorem 2.2] we have

Lemma 3.1. *$F_q(\lambda)$ has only simple zeros.*

Proof. Multiplying the equation in (3.a) by z and the equation in (3.b) by φ and subtracting we find that

$$\varphi'' z - \varphi z'' = [\varphi' z - \varphi z']' = p^2\varphi^2.$$

Integration gives

$$\varphi'(1, \lambda)z(1, \lambda) - \varphi(1, \lambda)z'(1, \lambda) = \int_0^1 p^2(x)\varphi^2(x, \lambda)\, dx > 0.$$

Thus the two equations $F_q(\lambda) = c\varphi(1, \lambda) + d\varphi'(1, \lambda) = 0$ and $\partial F_q(\lambda)/\partial\lambda = cz(1, \lambda) + dz'(1, \lambda) = 0$ cannot be satisfied simultaneously. □

The proof of Theorem 1.1 will be completed by using standard techniques to estimate the growth of solutions of the initial value problems

$$-[1/p^2(x)]D^2\varphi_j(x) + q(x)\varphi_j(x) = \lambda\varphi_j(x),$$
$$\varphi_j^{(i-1)}(0, \lambda) = \delta_{ij}, \qquad i, j, = 1, 2.$$

The estimates are based on the solutions $y_1(x, \lambda)$, $y_2(x, \lambda)$ previously developed for the case $q(x) = 0$. We begin with the variation of parameters representations

$$\varphi_j(x, \lambda) = y_j(x, \lambda) + \int_0^x [y_1(t, \lambda)y_2(x, \lambda) - y_1(x, \lambda)y_2(t, \lambda)]p^2(t)q(t)\varphi_j(t, \lambda)\, dt,$$

$$\varphi_j'(x, \lambda) = y_j'(x, \lambda) + \int_0^x [y_1(t, \lambda)y_2'(x, \lambda) - y_1'(x, \lambda)y_2(t, \lambda)]p^2(t)q(t)\varphi_j(t, \lambda)\, dt.$$

Let $\Phi(x, t, \lambda) = y_1(t, \lambda)y_2(x, \lambda) - y_1(x, \lambda)y_2(t, \lambda)$. Observe that for each t the function $\Phi(x, t, \lambda)$ satisfies the equation $-\Phi''(x, t, \lambda) = \lambda p^2(x)\Phi(x, t, \lambda)$ with the initial conditions $\Phi(t, t, \lambda) = 0$ and $(d/dx)\Phi(t, t, \lambda) = y_1(t, \lambda)y_2'(t, \lambda) - y_1'(t, \lambda)y_2(t, \lambda) = 1$ since it is just the Wronskian of the two solutions y_1, y_2.

Now if $x_n \le t < x \le x_{n+1}$ then $\Phi(x, t, \lambda) = \sin(p_n\omega[x - t])/[p_n\omega]$. On the other hand, if $x_n \le t < x_{n+1} < x$ then the solution crosses one of the jumps, and after applying the appropriate transition matrix we find that

$$\begin{aligned}\Phi(x, t, \lambda) = {} & \sin(p_n\omega[x_{n+1} - t])\cos(p_{n+1}\omega[x - x_{n+1}])/[p_n\omega] \\ & + \cos(p_n\omega[x_{n+1} - t])\sin(p_{n+1}\omega[x - x_{n+1}])/[p_{n+1}\omega] \\ = {} & [1/2][1/p_n + 1/p_{n+1}]\sin(p_n\omega[x_{n+1} - t] + p_{n+1}\omega[x - x_{n+1}])/\omega \\ & + [1/2][1/p_n - 1/p_{n+1}]\sin(p_n\omega[x_{n+1} - t] - p_{n+1}\omega[x - x_{n+1}])/\omega.\end{aligned}$$

In these two cases we get respectively the estimates

$$|\Phi(x, t, \lambda)| \le \exp(|\operatorname{Im}(\omega)|p_n[x - t])$$

and

$$\begin{aligned}|\Phi(x, t, \lambda)| &\le [1/p_n + 1/p_{n+1}]\exp(|\operatorname{Im}(\omega)|(p_n[x_{n+1} - t] + p_{n+1}[x - x_{n+1}])) \\ &\le [1/p_n + 1/p_{n+1}]\exp(|\operatorname{Im}(\omega)|([p_n + p_{n+1}][x - t])).\end{aligned}$$

Using the representation (2.c) it is then straightforward to establish

Lemma 3.2. *There are constants* K, P *such that for all* $0 \le t \le x \le 1$ *we have*

$$|\Phi(x, t, \lambda)| \le K\exp(|\operatorname{Im}(\omega)|P[x - t]),$$
$$|(d/dx)\Phi(x, t, \lambda)| \le K\exp(|\operatorname{Im}(\omega)|P[x - t]),$$

and for $|\lambda| \ge 1$

$$|\Phi(x, t, \lambda)| \le K\exp(|\operatorname{Im}(\omega)|P[x - t])/|\omega|.$$

Similarly we have the estimates

Lemma 3.3. *There are constants* K, P *such that for all* $0 \le x \le 1$

$$|y_1(x, \lambda)| \le K\exp(|\operatorname{Im}(\omega)|Px),$$
$$|y_2(x, \lambda)| \le K\exp(|\operatorname{Im}(\omega)|Px),$$
$$|y_2'(x, \lambda)| \le \exp(|\operatorname{Im}(\omega)|Px),$$

and for $|\lambda| \ge 1$

$$|y_1'(x, \lambda)| \le K|\omega|\exp(|\operatorname{Im}(\omega)|Px),$$
$$|y_2(x, \lambda)| \le K\exp(|\operatorname{Im}(\omega)|Px)/|\omega|.$$

Based on these estimates, standard Picard iteration arguments [14, p. 331] can be used to derive estimates for $\varphi_j(x, \lambda)$ if $q(x) \in L^1[0, 1]$. For instance if $\psi_n(x, \lambda)$ is defined by

$$\psi_0(x, \lambda) = y_1(x, \lambda),$$
$$\psi_n(x, \lambda) = y_1(x, \lambda) + \int_0^x \Phi(x, t, \lambda)p^2(t)q(t)\psi_{n-1}(t, \lambda)\,dt,$$

one readily proves by induction that there are constants K and P such that

$$|\psi_n - \psi_{n-1}|(x, \lambda) \le \exp(|\operatorname{Im}(\omega)|P)K^{n+1}\left[\int_0^x |q(s)|\,ds\right]^n/n!$$

Such arguments provide

Lemma 3.4. *Suppose that* $q(x) \in L^1[0, 1]$. *The functions* $\varphi_j^{(i-1)}(x, \lambda)$, $i, j = 1, 2$, *are entire functions of* λ *for each* $x \in [0, 1]$, *with order* $\frac{1}{2}$. *There are constants* K, P *such that for* $|\lambda| \geq 1$

$$\begin{aligned}
|\varphi_1(x, \lambda) - y_1(x, \lambda)| &\leq K \exp(|\operatorname{Im}(\omega)|Px)/|\omega|, \\
|\varphi_1'(x, \lambda) - y_1'(x, \lambda)| &\leq K \exp(|\operatorname{Im}(\omega)|Px), \\
|\varphi_2(x, \lambda) - y_2(x, \lambda)| &\leq K \exp(|\operatorname{Im}(\omega)|Px)/|\lambda|, \\
|\varphi_2'(x, \lambda) - y_2'(x, \lambda)| &\leq K \exp(|\operatorname{Im}(\omega)|Px)/|\omega|.
\end{aligned}$$

Now we have all the ingredients needed for the proof of Theorem 1.1. Lemmas 3.1, 3.3, and 3.4 imply that

$$\begin{aligned}
F_q(\lambda) &= d\varphi(1, \lambda) - c\varphi'(1, \lambda) \\
&= -ac\varphi_1'(1, \lambda) + [da\varphi_1(1, \lambda) - bc\varphi_2'(1, \lambda)] + bd\varphi_2(1, \lambda)
\end{aligned}$$

is an entire function of order $\frac{1}{2}$, which is determined up to a nonzero scalar by the eigenvalues of problem (1.a). Suppose that $ac \neq 0$. By Lemma 3.4 if $\nu > 0$ then

$$\begin{aligned}
&\lim_{L\to\infty} [2/L] \int_1^L F_q(\lambda)\omega^{-1} \sin(\nu\omega)\, d\omega \\
&\quad = \lim_{L\to\infty} [2/L] \int_1^L -ac\varphi_1'(1, \lambda)\omega^{-1} \sin(\nu\omega)\, d\omega \\
&\quad = \lim_{L\to\infty} [2/L] \int_1^L -acy_1'(1, \lambda)\omega^{-1} \sin(\nu\omega)\, d\omega \\
&\quad = \lim_{L\to\infty} [2/L] \int_1^L F_0(\lambda)\omega^{-1} \sin(\nu\omega)\, d\omega.
\end{aligned}$$

In short, the presence of $q(x)$ has no affect on these limits and the proof follows from the previously established case when $q(x) = 0$. If $ac = 0$, the needed data is computed, as appropriate, from

$$\lim_{L\to\infty} [2/L] \int_1^L F_q(\lambda) \cos(\nu\omega)\, d\omega = \lim_{L\to\infty} [2/L] \int_1^L F_0(\lambda) \cos(\nu\omega)\, d\omega$$

or from

$$\lim_{L\to\infty} [2/L] \int_1^L F_q(\lambda)\omega \sin(\nu\omega)\, d\omega = \lim_{L\to\infty} [2/L] \int_1^L F_0(\lambda)\omega \sin(\nu\omega)\, d\omega.$$

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