PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 120, Number 2, February 1994

ON THE RAMIFICATION THEORY OF REGULAR SCHEMES

ZHAOHUA LUO

(Communicated by Eric Friedlander)

ABSTRACT. We prove a fundamental theorem on the ramification of morphisms of regular schemes, extending Dedekind's theorem of different.

Suppose $f: X \to Y$ is a dominant morphism of smooth varieties unramified at the generic point (e.g., a birational morphism). Classically the ramification divisor E_f of f is defined via canonical sheaves, i.e., $E_f = \operatorname{div}(f^*(\omega_X \otimes \omega_Y^{-1}))$ (cf. [D]). If X, Y are regular schemes and f is essentially of finite type as above, then E_f can be defined to be the effective divisor determined by the Kähler different (sheaf) of the morphism f (cf. [L2]).

For any $x \in X$ and $y = f(x) \in Y$ we shall introduce three important invariants $r(xy)$, $e(xy)$, and $w(xy)$, where $r(xy)$ is the multiplicity of E_f at x and $e(xy)$ is the supremum of the multiplicities at x of the products of local coordinates of Y at y. The main purpose of this paper is to prove the following theorem, which is a generalization of the main theorem of ramification theory of algebraic number theory (due to Dedekind).

Theorem 1 (Geometric form). *Let $f: X \to Y$ be a morphism of regular schemes as above. Then for any $x \in X$ and $y = f(x)$ we have*

$$(*)\qquad \begin{aligned} r(xy) &\geq e(xy) - \operatorname{codim} x \geq w(xy) + \operatorname{codim} y - \operatorname{codim} x \\ &\geq \operatorname{codim} y - \operatorname{codim} x \geq 0. \end{aligned}$$

(i) *Suppose f is a finite morphism and* $\operatorname{codim} x = \operatorname{codim} y = 1$. *Then* $r(xy) = e(xy) - 1$ *if and only if the residue field $k(x)$ of x is a separable extension of the residue field $k(y)$ of y and $e(xy)$ is not a multiple of the characteristic of $k(y)$.*

(ii) *Suppose f is a birational morphism. Then* $r(xy) = \operatorname{codim} y - \operatorname{codim} x$ *if and only if x and y determine the same discrete valuation of the rational function field; in this case we have $m_y\mathcal{O}_x = m_x$ (thus y is the generic point of a component of $f^{-1}(x)$).*

The expression $(*)$ is one of the most important formulas for the ramification theory of regular schemes (see [L, L1, L3] for applications); a preliminary

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Received by the editors March 12, 1992.
1991 *Mathematics Subject Classification*. Primary 14E22, 13H05.
Supported partially by the Tian Yuan Foundation.

version of $(*)$ (i.e., $r(AB) \geq e(AB) - 1$ for the case $\operatorname{codim} x = 1$) was announced in [L1] and proved in [L2].

Since the ramification theory of regular schemes is essentially a local problem, we shall carry out our study mainly in the category of commutative algebras. Theorem 1 is proved in its algebraic form at the end of the paper. We first introduce some notation and terminology.

In this paper by a pair (A, B) of regular local rings we always mean a dominating pair $A \supseteq B$ of regular local rings such that the quotient field $Q(A)$ of A is a finite separable extension of the quotient field $Q(B)$ of B and A is a quotient ring of a finitely generated B-algebra (i.e., A is a B-algebra essentially of finite type). Denote by M and N the maximal ideals of A and B respectively.

Denote by $D(A/B)$ the differential module of A over B. $D(A/B)$ is a finitely generated torsion A-module and the 0-Fitting ideal $d(A/B)$ of $D(A/B)$ is a nonzero principal ideal of A.

If P is any subset of a local ring A, we define

$$\operatorname{ord}_A(P) = \sup\{t \in \mathbb{Z} | P \subset M^t\} \in \mathbb{Z} \cup \infty.$$

If A is a regular local ring then ord_A determines a discrete valuation v_A of the quotient field $Q(A)$ of A; denote by $Z(A)$ the discrete valuation ring of v_A. If $E = \{b_1, \dots, b_s\}$ is a finite subset of A, for simplicity we shall write $I(E)$ for the product $b_1 \cdot b_2 \cdots b_s$ of these elements.

Definition 1. Suppose (A, B) is a pair of regular local rings with $\dim A = m$ and $\dim B = n$ (here $\dim A = \operatorname{krull}\dim A$). We define

$$\begin{aligned}
r(AB) &= v_A(d(A/B));\\
e(AB) &= \sup\{v_A(I(E)) | E \text{ is a minimal basis of } N\};\\
w(AB) &= \sup\{t(v_A(Q) - 1) | Q \text{ is a prime ideal of } B \text{ of height } t\\
&\qquad\qquad \text{such that } B/Q \text{ is regular}\};\\
s(AB) &= (n - m)v_A(N).
\end{aligned}$$

$r(AB)$ and $e(AB)$ are called the ramification index and the reduced ramification index of A over B respectively. (Clearly we have $e(AB) - m \geq w(AB) + n - m \geq s(AB) \geq n - m \geq 0$. We shall prove that $e(AB) < \infty$.)

Let $\mathscr{C}(A/B)$ be the set of prime ideals of A of height 1 such that P is ramified over B (i.e., A_P is ramified over $B_{P \cap B}$); $\mathscr{C}(A/B)$ is a finite set.

The reader is referred to [K1, K2, L2] for the general theory of Kähler different.

In the following we assume that (A, B), (B, C), and (C, D) are pairs of regular local rings (then (A, C), (A, D), and (B, D) are also pairs of regular local rings).

Lemma 1. *$r(AB) = 0$ if and only if A is unramified over B. If $Q(A) = Q(B)$, then $r(AB) = 0$ if and only if $A = B$.*

Proof. A is unramified if and only if $d(A/B) = (1)$, i.e., $r(AB) = 0$ (see [K1]). If $Q(A) = Q(B)$ then A is unramified over B if and only if $A = B$; thus $r(AB) = 0$ if and only if $A = B$ in this case.

Lemma 2. $r(AB) = \sum_{P\in\mathscr{C}(A/B)} r(A_P B_{P\cap B})(v_A(P))$. *Thus* $r(AB) \geq |\mathscr{C}(A/B)|$.

Proof. Since $d(AB)$ is a principal nonzero ideal of A, $d(AB) = \prod a_i^{t_i}$ with $a_i \in A$ irreducible. Then $\mathscr{C}(A/B)$ consists of all the prime ideals $P_i = (a_i)A$. We have $v_A(P_i) = v_A(a_i)$ and $r(A_{P_i}/B_{P_i\cap B}) = t_i$. Thus $r(AB) = v_A(d(A/B)) = \sum(v_A(a_i))t_i = \sum_{P\in\mathscr{C}(A/B)} r(A_P B_{P\cap B})(v_A(P))$.

Lemma 3. *Write* $r(ABC) = v_A(d(B/C))$. *Then we have* $r(AC) = r(AB) + r(ABC)$. *Also* $r(ABC) = \sum_{P\in\mathscr{C}(B/C)} r(B_P C_{P\cap C})(v_A(P))$.

Proof. We have $d(A/C) = d(A/B)(d(B/C)A)$ [L2]; thus $r(AC) = r(AB) + r(ABC)$. The proof of the second assertion is similar to that of Lemma 2.

Lemma 4. $r(ABC) \geq r(BC)$ *and* $r(Z(B)BC) = r(BC)$. *Thus* $r(AC) \geq r(AB) + r(BC)$.

Proof. Note that $v_A(P) \geq v_B(P)$ for any subset P of B and $v_{Z(B)} = v_B$.

Lemma 5. $r(ABD) = r(ABC) + r(ACD)$ *where* $r(ABD)$ *and* $r(ACD)$ *have the same meaning as* $r(ABC)$.

Proof. We have $r(ABD) = r(AD) - r(AB) = (r(AC) + r(ACD)) - r(AB) = (r(AC) - r(AB)) + r(ACD) = r(ABC) + r(ACD)$.

Lemma 6. *Suppose* A *is a discrete valuation ring and* B *is the first quadratic transform of* C *along* A (*see* [A]). *Then* $e(AC) \leq e(AB) + s(AC)$ *and* $r(ABC) = s(AC)$.

Proof. Let $E = \{b_1, \ldots, b_t\}$ be a minimal basis of the maximal ideal Q of C. Suppose we have arranged the b_i so that $v_A(b_1) = \cdots = v_A(b_{t'}) < v_A(b_{t'+1}) \leq \cdots \leq v_A(b_t)$. Then $v_A(Q) = v_A(b_1)$, which implies that $s(AC) = (t-1)v_A(b_1)$. The set $E' = \{b_1, b_{t'+1}/b_1, \ldots, b_t/b_1\}$ is part of a minimal basis of the maximal ideal N of B; hence $e(AB) \geq v_A(I(E'))$. Since $v_A(b_i) = v_A(b_1)$ for $i \leq t'$, we have $v_A(I(E)) = v_A(I(E')) + (t-1)v_A(b_1) \leq e(AB) + s(AC)$ for any minimal basis E of Q. It follows that $e(AC) \leq e(AB) + s(AC)$. As B is a quadratic transform of C, we have $d(B/C) = Q^{t-1}B$ (cf. [L2]); hence $r(ABC) = v_A(d(B/C)) = (t-1)v_A(Q) = s(AC)$.

Suppose $Q(A) = Q(B)$ and A is a discrete valuation ring. Consider the quadratic sequence along A starting from B such that each B_i is the first quadratic transform of B_{i-1} along v_A (cf. [A, p. 336]):

$$A = B_t \supset \cdots \supset B_2 \supset B_1 \supset B_0 = B.$$

The existence of such a finite sequence is guaranteed by Lemma 3, as the length t of any such strictly descending chain between A and B is bounded by $r(AB)$, because we have $r(AB) = \sum_{i=0,\ldots,t-1} r(AB_{i+1}B_i) \geq t$.

Lemma 7. *Suppose* $Q(A) = Q(B)$ *and* A *is a discrete valuation ring. Then*

(i) $r(AB) = \sum_{i=0,\ldots,t-1} s(AB_i) \geq e(AB) - 1$.

(ii) *The following assertions are equivalent*:

(a) $A = Z(B)$;

(b) $r(AB) = s(AB)$;

(c) $r(AB) = n - 1$.

Proof. (i) Note that $s(AB_i) = r(AB_{i+1}B_i)$ by Lemma 6; hence $r(AB) = \sum_{i=0,\ldots,t-1} r(AB_{i+1}B_i) = \sum_{i=0,\ldots,t-1} s(AB_i)$. Applying Lemma 6 and using induction on t we see that $e(AB)$ is finite. Also by that lemma we obtain $r(AB) = \sum_{i=0,\dots,t-1} s(AB_i) \geq \sum_{i=0,\dots,t-1} e(AB_i) - e(AB_{i+1}) = e(AB) - e(AB_t) = e(AB) - 1$.

(ii) Write $c = \sum_{i=1,\dots,t-1} s(AB_i)$. Then $c = 0$ if and only if $A = Z(B)$. But $r(AB) = c + s(AB) \geq c + n - 1$. Thus $c = 0$ if and only if $r(AB) = s(AB)$. Thus (a) and (b) are equivalent. If $r(AB) = n-1$ then $c = 0$; thus $A = Z(B)$. Clearly $A = Z(B)$ implies $r(AB) = n - 1$. Thus (a) and (c) are equivalent.

Remark. The formula $r(AB) = \sum_{i=0,\dots,t-1} s(AB_i)$ gives another definition of $r(AB)$ for any birational pair (A, B) of regular local rings such that $\dim A = 1$, without referring to Kähler different. If A and B are regular localities of an algebraic function field K/k then one can use this formula to prove that our definition of $r(AB)$ coincides with the usual one obtained by means of the canonical divisors of K/k (see [L2]).

Lemma 8. $r(AB) = r(Z(A)B) - m + 1$.

Proof. We have $r(AB) = r(Z(A)AB) = r(Z(A)B) - r(Z(A)A) = r(Z(A)B) - (m-1)$ by Lemmas 3, 4, and 7(ii).

Theorem 2 (Algebraic form). *For any pair (A, B) of regular local rings we have*

$$(\#) \qquad \begin{aligned} r(AB) &\geq e(AB) - \dim A \geq w(AB) + \dim B - \dim A \\ &\geq s(AB) \geq \dim B - \dim A \geq 0. \end{aligned}$$

(i) *Suppose A, B are discrete valuation rings; then $r(AB) = e(AB) - 1$ if and only if $e(AB)$ is not a multiple of* $\operatorname{ch} B/N$, *and A/M is a separable extension of B/N.*

(ii) *Suppose $Q(A) = Q(B)$; then $r(AB) = 0$ if and only if $A = B$, and $r(AB) = \dim B - \dim A$ if and only if $Z(A) = Z(B)$. If $Z(A) = Z(B)$ then $NA = M$.*

Proof. (i) If A, B are discrete valuation rings, then $d(AB)$ coincides with the Dedekind different of A over B (see [L2]), and $r(AB)$ is called the differential exponent in [ZS]. Thus (i) is the content of Dedekind's theorem of different.

To prove (#) we consider $r(AB) = r(Z(A)B) - m + 1$ (Lemma 8). If we can prove $r(Z(A)B) \geq e(Z(A)B) - 1$, then, since $e(Z(A)B) = e(AB)$, we would obtain $r(AB) \geq e(AB) - m$. Thus we may assume that A is a discrete valuation ring. Let $A_0 = A \cap Q(B)$, and put $h = e(AA_0)$. Then we have $r(AA_0) \geq h - 1$ by (i) and $r(A_0B) \geq e(A_0B) - 1$ by Lemma 7. Thus $r(AB) = r(AA_0) + hr(A_0B) \geq h - 1 + h(e(A_0B) - 1) = h(e(A_0B)) - 1 = e(AB) - 1$. (It is easy to see that $e(AB) = h(e(A_0B))$.) This proves (#).

We now prove (ii). In the following we assume $Q(A) = Q(B)$. We have $r(AB) = r(Z(A)B) - m + 1$ (Lemma 8); thus $r(AB) = n - m$ if and only if $r(Z(A)B) = n - 1$, which is equivalent to $Z(A) = Z(B)$ by Lemma 7.

Finally we assume $Z(A) = Z(B)$. The assertion that $NA = M$ was proved in [S]. We give a simple algebraic proof based upon the inequality $r(AB) \geq e(AB) - \dim A$ obtained above.

Let $E = \{b_1, \dots, b_n\}$ and $E' = \{a_1, \dots, a_m\}$ be the minimal bases of N and M respectively. We can choose a_i such that $a_t, \dots, a_m$ generates $NA \bmod M^2$; then $NA = M$ if and only if $t = 1$. Let A' be the localization of $A[a_2/a_1, \dots, a_m/a_1]$ at the maximal ideal generated by $E_0 =$

$\{a_1, a_2/a_1, \dots, a_m/a_1\}$. Then (A', A) is a pair of regular local rings. We calculate $r(A'B)$ and $e(A'B)$.

(a) We have $r(A'B) = r(A'A) + r(A'AB) = m - 1 + v_{A'}(d(A/B))$ where $d(A/B) = (a)A$ with $v_A(a) = r(AB) = n - m$. It is easy to see that $v_{A'}(g) \leq 2v_A(g)$ for any $g \in M$ (using the fact that E_0 is a minimal basis of the maximal ideal of A'). Thus $r(A'B) \leq m - 1 + 2(n - m) = 2n - m - 1$.

(b) We have $v_{A'}(a_i) = 2$ for any $1 < i \leq m$. Each b_i can be written as a linear combination of $a_t, \dots, a_m$ mod M^2. Thus if $t > 1$ then $v_{A'}(b_i) \geq 2$ for all i, which implies $e(A'B) \geq 2n$. Thus $e(A'B) - m \geq 2n - m > 2n - m - 1 = r(A'B)$, contradicting the inequality $r(A'B) \geq e(A'B) - m$ of (#). Thus $t = 1$, which means $NA = M$. The proof of the theorem is complete.

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ACADEMIA SINICA, INSTITUTE OF SYSTEMS SCIENCE, BEIJING 100080, PEOPLE'S REPUBLIC OF CHINA

Current address: 24989 Prospect Avenue, Loma Linda, California 92354