ON THE ISOMETRIES OF $H_E^\infty(B)$

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ABSTRACT. Let E be a complex Banach space on which all the multipliers are trivial. Let $H_E^\infty(B)$ denote the Banach space of E -valued bounded holomorphic functions on the open unit ball B of $\mathbb{C}^n$. In this paper we prove that every linear isometry T of $H_E^\infty(B)$ onto itself is of the form $(TF)(z) = \mathfrak{T}F(\varphi(z))$ for all $F \in H_E^\infty(B)$, $z \in B$, where $\mathfrak{T}$ is a linear isometry of E onto itself and φ is a biholomorphic map of B .

1. INTRODUCTION

Let $n \geq 1$ be a fixed integer. Throughout this paper the letters B and E will denote the open unit ball of the complex n -dimensional Euclidean space $\mathbb{C}^n$ and a complex Banach space, respectively. $H_E^\infty(B)$ denotes the Banach space of all E -valued bounded holomorphic functions defined on B provided with the supremum norm. (For the definition of E -valued holomorphic functions, see [5, §3.15].) A bounded operator A on E is called a *multiplier* if there is a positive number ρ such that, for given $e_1, e_2 \in E$ and $r > 0$, the assumption $\|e_1 - \lambda e_2\| \leq r$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \leq \rho$ implies $\|e_1 - Ae_2\| \leq r$. (For equivalent definitions of multipliers, see [2, Part I, 3.A].) $\text{Mult}(E)$ stands for the set of all multipliers on E .

In the case of the dimension $n = 1$, B coincides with the open unit disc D in the complex plane $\mathbb{C}$. Cambern and Jarosz [3] have recently proved the following theorem:

Theorem A. *Let E be a Banach space with $\text{Mult}(E) = \mathbb{C}$, and let T be a linear isometry of $H_E^\infty(D)$ onto itself. Then T is of the form*

$$(TF)(z) = \mathfrak{T}F(\tau(z)), \quad F \in H_E^\infty(D), \quad z \in D,$$

where $\mathfrak{T}$ is a linear isometry of E onto itself and τ is a conformal map of D onto itself.

In place of the assumption $\text{Mult}(E) = \mathbb{C}$, by assuming that E is uniformly convex and uniformly smooth, Lin [7] has obtained the same result as Theorem A, via a different method. If E is uniformly convex, then it is strictly convex. (For these notions of Banach spaces, see, e.g., [4].) If E is strictly convex, then $\text{Mult}(E) = \mathbb{C}$ [6, Theorem 12.7]. Hence Theorem A contains Lin's result. The

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purpose of this paper is to prove a generalization of Theorem A to the case of any dimension $n \geq 1$.

Let $H^\infty(B)$ denote the Banach space of all bounded holomorphic complex functions on B provided with the supremum norm. $\text{Aut}(B)$ stands for the set of all biholomorphic maps of B . For a characterization of the isometries of $H^\infty(B)$, we refer to [8, §7.5.7] (see also [1]):

Theorem B. *If T is a linear isometry of $H^\infty(B)$ onto itself, then there exist a $\varphi \in \text{Aut}(B)$ and a $c \in \mathbb{C}$ with $|c| = 1$, such that*

$$Tf = cf \circ \varphi, \quad f \in H^\infty(B).$$

The main result of the present paper is the following:

Theorem 1. *Let E be a Banach space with $\text{Mult}(E) = \mathbb{C}$. If T is a linear isometry of $H_E^\infty(B)$ onto itself, then there exist a $\varphi \in \text{Aut}(B)$ and a linear isometry $\mathfrak{T}$ of E onto itself such that*

$$(TF)(z) = \mathfrak{T}F(\varphi(z))$$

for all $F \in H_E^\infty(B)$ and $z \in B$.

In order to prove Theorem 1 we argue along the lines of [3], but we need to solve Gleason's problem for the space $H_E^\infty(B)$. In §2 we describe this subject. The last section is devoted to the proof of Theorem 1.

2. GLEASON'S PROBLEM FOR $H_E^\infty(B)$

To begin with, we recall the Ahern-Schneider solution of Gleason's problem for the space $H^\infty(B)$. Let $S = \partial B$ and σ denote the unit sphere of $\mathbb{C}^n$ and the rotation invariant positive Borel measure on S for which $\sigma(S) = 1$, respectively. For $a = (a_1, \dots, a_n) \in B$, $z = (z_1, \dots, z_n) \in B$, $\zeta = (\zeta_1, \dots, \zeta_n) \in S$, $j \in \{1, \dots, n\}$, and $f \in H^\infty(B)$, we define

$$K_j^a(z, \zeta) = \frac{\overline{\zeta_j} \{C(z, \zeta) - C(a, \zeta)\}}{\langle z - a, \zeta \rangle}$$

where $\langle z - a, \zeta \rangle = \sum_{j=1}^n (z_j - a_j) \overline{\zeta_j}$ and $C(z, \zeta) = (1 - \langle z, \zeta \rangle)^{-n}$ (the Cauchy kernel for B). Moreover, we define

$$(L_j^a f)(z) = \int_S K_j^a(z, \zeta) f(\zeta) d\sigma(\zeta).$$

The Ahern-Schneider solution of Gleason's problem for $H^\infty(B)$ is the following [8, §6.6.2]:

Theorem C. *If $f \in H^\infty(B)$ and $a \in B$, then*

$$L_j^a f \in H^\infty(B), \quad \|L_j^a f\|_\infty \leq c \|f\|_\infty \quad (j = 1, \dots, n)$$

and

$$f(z) - f(a) = \sum_{j=1}^n (z_j - a_j) (L_j^a f)(z), \quad z \in B,$$

where c is a positive constant depending only on the point a and the dimension n .

Using this theorem, we solve Gleason's problem for the space $H_E^\infty(B)$:

Theorem 2. Suppose $F \in H_E^\infty(B)$ and $a \in B$. Then the following hold:

(i) For any $z \in B$ and any integer j with $1 \leq j \leq n$, the E -valued Bochner integral

$$(1) \quad (\mathcal{L}_j^a F)(z) = \int_S K_j^a(z, \zeta) F(\zeta) d\sigma(\zeta)$$

exists.

(ii) $\mathcal{L}_j^a F \in H_E^\infty(B)$ ($j = 1, \dots, n$).

(iii) $F(z) - F(a) = \sum_{j=1}^n (z_j - a_j)(\mathcal{L}_j^a F)(z)$, $z \in B$.

Proof. For an E -valued function G defined on a set X and $\Lambda \in E^*$ (the dual space of E), we define the scalar function ΛG by

$$(\Lambda G)(x) = \Lambda(G(x)), \quad x \in X.$$

Fix $z \in B$ and $j \in \{1, \dots, n\}$. Put

$$G_j^z(\zeta) = K_j^a(z, \zeta) F(\zeta)$$

for almost all $\zeta \in S$ ($[\sigma]$). Then

$$(\Lambda G_j^z)(\zeta) = K_j^a(z, \zeta)(\Lambda F)(\zeta)$$

for any $\Lambda \in E^*$ and almost all $\zeta \in S$ ($[\sigma]$). The assumption $F \in H_E^\infty(B)$ implies $\Lambda F \in H^\infty(B)$. Thus we have $\Lambda G_j^z \in L^1(\sigma)$ and

$$\int_S (\Lambda G_j^z) d\sigma = (L_j^a(\Lambda F))(z).$$

Moreover, it holds that

$$(2) \quad \int_S \|K_j^a(z, \zeta) F(\zeta)\| d\sigma \leq \|F\|_\infty \int_S |K_j^a(z, \zeta)| d\sigma(\zeta) < \infty.$$

By (2) and the fact that the open subset B of $\mathbb{C}^n$ is separable, the E -valued Bochner integral (1) exists. (See [5, Theorems 3.5.3 and 3.7.4].) This proves (i).

By (1) and Theorem C, we have

$$\Lambda(\mathcal{L}_j^a F) = L_j^a(\Lambda F) \in H^\infty(B),$$

$$\|\Lambda(\mathcal{L}_j^a F)\|_\infty \leq c \|\Lambda F\|_\infty \leq c \|\Lambda\| \|F\|_\infty,$$

and

$$\begin{aligned} (\Lambda F)(z) - (\Lambda F)(a) &= \sum_{j=1}^n (z_j - a_j) \Lambda(\mathcal{L}_j^a F)(z) \\ &= \Lambda \left(\sum_{j=1}^n (z_j - a_j) (\mathcal{L}_j^a F)(z) \right) \end{aligned}$$

for any $\Lambda \in E^*$. Hence

$$F(z) - F(a) = \sum_{j=1}^n (z_j - a_j) (\mathcal{L}_j^a F)(z).$$

Moreover, it follows that

$$\begin{aligned}\|\mathfrak{L}_j^a F\|_\infty &= \sup\{\|(\mathfrak{L}_j^a F)(z)\| : z \in B\} \\ &= \sup\{|\Lambda((\mathfrak{L}_j^a F)(z))| : z \in B, \Lambda \in E^*, \|\Lambda\| \leq 1\} \\ &\leq \sup\{\|\Lambda(\mathfrak{L}_j^a F)\|_\infty : \Lambda \in E^*, \|\Lambda\| \leq 1\} \leq c\|F\|_\infty < \infty.\end{aligned}$$

This shows $\mathfrak{L}_j^a F \in H_E^\infty(B)$ ($j = 1, \dots, n$). The proof is complete.

3. PROOF OF THEOREM 1

Before we give the proof of Theorem 1, we establish two propositions. The following first one is almost trivial, but for the sake of reference we state it:

Proposition 3. *Suppose $f \in H^\infty(B)$, and define the multiplication operator M_f on $H_E^\infty(B)$ by*

$$(M_f F)(z) = f(z)F(z), \quad F \in H_E^\infty(B), \quad z \in B.$$

Then $M_f \in \text{Mult}(H_E^\infty(B))$ and $\|M_f\| = \|f\|_\infty$.

Proposition 4 (cf. [3, Proposition 2]). *Let A be a multiplier on $H_E^\infty(B)$. Then for any $z \in B$ there is a multiplier A_z on E such that*

$$(AF)(z) = A_z(F(z)), \quad F \in H_E^\infty(B).$$

Proof. Fix $z \in B$. We define a map $A_z : E \rightarrow E$ by $A_z(e) = (A(1_B e))(z)$, where 1_B is the constant function 1 on B . Then we have $A_z \in \text{Mult}(E)$, as shown in the proof of [3, Proposition 2].

By Theorem 2, for any $F \in H_E^\infty(B)$ there exist $G_j \in H_E^\infty(B)$ ($j = 1, \dots, n$) such that

$$F(w) - F(z) = \sum_{j=1}^n (w_j - z_j) G_j(w), \quad w \in B.$$

Put $f_j(w) = w_j - z_j$ ($w \in B, j = 1, \dots, n$). Then $\{f_1, \dots, f_n\} \subset H^\infty(B)$ and

$$F - 1_B F(z) = \sum_{j=1}^n M_{f_j} G_j$$

in B . Since $\text{Mult}(H_E^\infty(B))$ is a commutative operator algebra [2, p. 54], it follows from Proposition 3 that

$$AF - A(1_B F(z)) = \sum_{j=1}^n M_{f_j}(AG_j)$$

in B . Since $f_j(z) = 0$ for $j = 1, \dots, n$, we conclude that

$$(AF)(z) = A(1_B F(z))(z) = A_z(F(z)).$$

This completes the proof.

The following corollary to Proposition 4 is shown by the same way used in [3, the proof of the corollary, p. 465].

Corollary 5. Suppose $\text{Mult}(E) = \mathbb{C}$. Then each $A \in \text{Mult}(H_E^\infty(B))$ is of the form $A = M_f$, where f is a function in $H^\infty(B)$.

Now we can prove the main result of the present paper:

Proof of Theorem 1. Suppose $\text{Mult}(E) = \mathbb{C}$ and T is a linear isometry of $H_E^\infty(B)$ onto itself. Let $f \in H^\infty(B)$. Then it is easy to verify that $TM_fT^{-1} \in \text{Mult}(H_E^\infty(B))$. By Corollary 5, we have $M_{\Phi(f)} = TM_fT^{-1}$ for some $\Phi(f) \in H^\infty(B)$. Since T is a linear isometry of $H_E^\infty(B)$ onto itself, it follows that Φ is a linear isometry of $H^\infty(B)$ onto itself and $\Phi(1_B) = 1_B$. By Theorem B, there exists a $\varphi \in \text{Aut}(B)$ such that

$$\Phi(f) = f \circ \varphi, \quad f \in H^\infty(B).$$

We define two linear isometries T_1 and T_2 of $H_E^\infty(B)$ onto itself by

$$(T_1F)(z) = F(\varphi^{-1}(z)), \quad F \in H_E^\infty(B), \quad z \in B,$$

and $T_2 = T_1 \circ T$. Then it follows that $T_2M_f = M_fT_2$ for any $f \in H^\infty(B)$.

Fix $z \in B$. Suppose $F \in H_E^\infty(B)$, $G \in H_E^\infty(B)$, and $F(z) = G(z)$. Since $F - G \in H_E^\infty(B)$, and $(F - G)(z) = 0$, Theorem 2 gives that there exist $P_j \in H_E^\infty(B)$ ($j = 1, \dots, n$) such that

$$F - G = \sum_{j=1}^n M_{f_j}P_j,$$

where $f_j(w) = w_j - z_j$ ($w \in B$, $j = 1, \dots, n$). Hence

$$T_2F - T_2G = \sum_{j=1}^n T_2(M_{f_j}P_j) = \sum_{j=1}^n f_j(T_2P_j).$$

Since $f_j(z) = 0$ ($j = 1, \dots, n$), we have

$$(T_2F)(z) = (T_2G)(z).$$

We can therefore define a map $\mathfrak{T}_z$ from E to E by

$$\mathfrak{T}_z(F(z)) = (T_2F)(z), \quad F \in H_E^\infty(B).$$

The above discussion shows that $\mathfrak{T}_z$ is a well-defined linear map of E to itself. Moreover, we can see that $\mathfrak{T}_z$ is, in fact, equal to a constant linear isometry $\mathfrak{T}$ of E onto itself, for all $z \in B$. This is shown by the same argument used in the last paragraph of the proof of [3, Proposition 3].

For $F \in H_E^\infty(B)$ and $z \in B$, we thus have

$$(TF)(z) = ((T_1^{-1} \circ T_2)F)(z) = T_1^{-1}(\mathfrak{T}(F(z))) = \mathfrak{T}(F(\varphi(z))).$$

This completes the proof of Theorem 1.

REFERENCES

1. P. R. Ahern and R. Schneider, *Isometries of H^∞* , Duke Math. J. **42** (1975), 321–326.
2. E. Behrends, *M-structure and the Banach-Stone Theorem*, Lecture Notes in Math., vol. 736, Springer-Verlag, Berlin, 1979.
3. M. Cambern and K. Jarosz, *Multipliers and isometries in H_E^∞* , Bull. London Math. Soc. **22** (1990), 463–466.

4. J. Diestel, *Geometry of Banach spaces-selected topics*, Lecture Notes in Math., vol. 485, Springer-Verlag, Berlin, 1975.
5. E. Hille and R. S. Phillips, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ., vol. 31, Amer. Math. Soc., Providence, RI, 1957.
6. K. Jarosz, *Perturbations of Banach algebras*, Lecture Notes in Math., vol. 1120, Springer-Verlag, Berlin, 1985.
7. P.-K. Lin, *The isometries of $H^\infty(E)$* , Pacific J. Math. **143** (1990), 69–77.
8. W. Rudin, *Function theory in the unit ball of $\mathbb{C}^n$* , Springer-Verlag, New York, 1980.

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