COMFORT TYPES OF ULTRAFILTERS

SALVADOR GARCIA-FERREIRA

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ABSTRACT. Let α be an infinite cardinal. Comfort pre-order on $\beta(\alpha) \setminus \alpha$ is defined as follows: for $p, q \in \beta(\alpha) \setminus \alpha$, $p \leq_c q$ if every q -compact space is p -compact. For $p \in U(\alpha)$, we let $T_{RK}(p)$ be the type of p and $T_c(p) = \{q \in U(\alpha) : q \leq_c p \leq_c q\}$. Since $T_c(p)$ is a union of types, it is natural to define $c_p = |\{T_{RK}(q) : q \in T_c(p)\}|$. It is evident that $\omega \leq c_p \leq 2^\alpha$ for $p \in U(\alpha)$. We show that if $p \in U(\alpha)$ then $|T_{RK}(p)| = |\alpha/p|$, and we use this equality to prove that $c_p = 2^\alpha$ whenever p is decomposable. We also note that if p is countably incomplete then $2^\omega \leq c_p \leq 2^\alpha$; if p is RK-minimal (selective) and $\omega < \alpha$ then $c_p = \omega$ and $c_p = 2^{\aleph_n}$ for each $p \in U(\aleph_n)$ and for $n < \omega$. Finally, we prove that, if α is a strong limit and $p \in U(\alpha)$ is indecomposable, then $\beta_p(\alpha)$ is a p -compact, noninitially α -compact space, where $\beta_p(\alpha)$ is the p -compactification of α .

1. PRELIMINARIES

Throughout this paper all spaces are assumed to be completely regular Hausdorff. If $f: X \rightarrow Y$ is a continuous function then $\bar{f}: \beta(X) \rightarrow \beta(Y)$ denotes the Stone extension of f . The remainder of X is the space $X^* = \beta(X) \setminus X$. The Greek letters α, γ , and κ will stand for infinite cardinal numbers, and ξ, ζ , and η for ordinal numbers. If α is a cardinal then also α stands for the discrete space of cardinality α , and $\beta(\alpha)$ is identified with the set of ultrafilters on α . For $A \subseteq \alpha$, the closure of A in $\beta(\alpha)$ is the set $\hat{A} = \{p \in \beta(\alpha) : A \in p\}$ and $A^* = \hat{A} \setminus A$. For $p \in \alpha^*$, the norm of p is $\|p\| = \min\{|A| : A \in p\}$. $U(\alpha) = \{p \in \alpha^* : \|p\| = \alpha\}$ is the set of uniform ultrafilters on α and $N(\alpha) = \beta(\alpha) \setminus U(\alpha)$. The Rudin-Keisler order on α^* is defined by $p \leq_{RK} q$ if there is $f \in {}^\alpha \alpha$ such that $\bar{f}(q) = p$, for $p, q \in \alpha^*$. If $p, q \in \alpha^*$, then $p \approx_{RK} q$ means that $p \leq_{RK} q$ and $q \leq_{RK} p$. It is not hard to prove that $p \approx_{RK} q$ iff there is a bijection σ of α such that $\bar{\sigma}(p) = q$. For $p \in \alpha^*$, the type of p is $T_{RK}(p) = \{q \in \alpha^* : p \approx_{RK} q\}$. Recall that $p \in U(\alpha)$ is *decomposable* if $\forall$ infinite $\lambda < \alpha \exists r \in U(\lambda)$ ($r \leq_{RK} p$), p is *indecomposable* if whenever $\omega < \lambda < \alpha$ there is no $r \leq_{RK} p$ such that $r \in U(\lambda)$, and p is *countably incomplete* if there is $q \in \omega^*$ such that $q \leq_{RK} p$. We say $f \in {}^\alpha \beta(\alpha)$ is a *strong embedding* if

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there is a partition $\{A_\xi: \xi < \alpha\}$ of α such that $f(\xi) \in \widehat{A}_\xi$ for each $\xi < \alpha$. For $p, q \in \alpha^*$, the tensor product of p and q is defined by

$$p \otimes q = \{A \subseteq \alpha \times \alpha: \{\xi < \alpha: \{\zeta < \alpha: (\xi, \zeta) \in A\} \in q\} \in p\}$$

(for other properties not considered here and background on tensor products the reader may consult [CN]). If $p, q \in \alpha^*$, then $p \otimes q$ is an ultrafilter on $\alpha \times \alpha$ and it can be viewed as a point in α^* via a bijection between α and $\alpha \times \alpha$. It is not hard to see that $\otimes$ is not an associative operation on α^* . Nevertheless, $\otimes$ induces a semigroup operation on the set of types $\{T_{RK}(p): p \in \alpha^*\}$. Hence, p^n stands for an arbitrary point in $T_{RK}(p)^n$, for $p \in \alpha^*$ and for $1 \leq n < \omega$.

The following lemma is a summary of basic results about the Rudin-Keisler order (proofs are available in [CN] or [G1]).

1.1. **Lemma.** *Let $p, q \in \alpha^*$. Then:*

- (1) *if $f \in \alpha^\alpha$, then $\bar{f}(p) \approx_{RK} p \Leftrightarrow \exists A \in p \ (f|_A \text{ is one-to-one})$;*
- (2) *(Blass) if $f \in {}^\alpha(T_{RK}(q))$ is a strong embedding, then $\bar{f}(p) \approx_{RK} p \otimes q$;*
- (3) *$p <_{RK} p \otimes q$ and $q <_{RK} p \otimes q$;*
- (4) *(M. E. Rudin) if $r \otimes q \approx_{RK} r \otimes p$ for some $r \in \alpha^*$, then $p \approx_{RK} q$.*

The following concept was introduced, in the context of nonstandard analysis, by Bernstein [B] and is a generalization of compactness.

1.2. **Definition** (Bernstein). Let $p \in \alpha^*$. A space X is p -compact if

$$\bar{f}(p) \in X \quad \text{for each } f \in {}^\alpha X.$$

Clearly, every compact space is p -compact for each $p \in \alpha^*$, and it is easy to prove that the product of a set of p -compact spaces is p -compact and p -compactness is closed-hereditary. Thus, for a space X , by a result proved by Herrlich and Van der Slot [HS], Franklin [F], and Woods [W], we have that $\beta_p(X) = \bigcap \{Y \subseteq \beta(X): X \subseteq Y, Y \text{ is } p\text{-compact}\}$ is the p -compact reflection of X for each $p \in \alpha^*$; Blass [Bl2] also introduced the sets $\beta_p(\omega)$, for $p \in \omega^*$, in a very natural way from recursion-theoretic considerations.

2. TYPES

The results of this section are essentially known, but we could not find them explicitly in the published literature. The main result is Theorem 2.3, where we estimate the cardinality of a type in terms of the cardinality of a suitable ultraproduct (for the definition of ultraproduct and a survey on ultraproducts see [CN]). We need the following

2.1. **Lemma** [Bl1]. *Let $f, g \in {}^\alpha \alpha$ such that f is one-to-one, and let $p \in \alpha^*$. Then $\bar{f}(p) = \bar{g}(p)$ if and only if $\{\xi < \alpha: f(\xi) = g(\xi)\} \in p$.*

Proof. $(\Leftarrow)$ This is evident.

$(\Rightarrow)$ Assume that $\bar{f}(p) = \bar{g}(p)$. Modifying f only on a set not in p , we can arrange that f is a bijection. Consider the function $f^{-1} \circ g \in {}^\alpha \alpha$. Applying [CN, Theorem 9.2(a)], we have that $\{\xi < \alpha: f^{-1}(g(\xi)) = \xi\} \in p$, since $\bar{f}^{-1} \circ \bar{g}(p) = p$. Hence, $\{\xi < \alpha: f(\xi) = g(\xi)\} \in p$.

The referee has provided the following example showing that the assumption that f and g are finite-to-one is not sufficient for Lemma 2.1. Fix $p \in \omega^*$,

and consider in ω^3 the sets $\{(x, y, z): (x, z) \in A\}$, $\{(x, y, z): (y, z) \in A\}$, and $\{(x, y, z): x \neq y\}$, where A ranges over $p \otimes p$. These sets have the finite intersection property, so there is an ultrafilter q on ω^3 containing them all. The projections to ω^3 defined by $f(x, y, z) = (x, z)$ and $g(x, y, z) = (y, z)$ are finite-to-one on $\{(x, y, z): x < z \text{ and } y < z\}$ which is in q , and $\bar{f}(q) = \bar{g}(q) = p \otimes p$, yet the set on which f and g agree is $\{(x, y, z): x = y\} \notin q$. Notice that, if p is RK-minimal in α^* (Q -point of ω^*) and $f, g \in {}^\alpha\alpha$ ($f, q \in {}^\omega\omega$ and f is finite-to-one), then the conclusion of Lemma 2.1 holds. The next theorem shows, assuming MA , there are points in ω^* which are non-RK-minimal and satisfy the conclusion of Lemma 2.1 without any additional condition on the functions.

2.2. Theorem. *Let $X = \{p_n: n < \omega\}$ and $\{r\}$ be RK-minimal points of ω^* such that r is not $\approx_{\text{RK}}$ -equivalent to p_n for $n < \omega$. If $p = \bar{e}(r)$ (where $e: \omega \rightarrow X$ is defined by $e(n) = p_n$ for $n < \omega$) and $f, g \in {}^\omega\omega$, then*

$$(*) \quad \bar{f}(p) = \bar{g}(p) \quad \text{iff} \quad \{n < \omega: f(n) = g(n)\} \in p.$$

In particular, if $p, q \in \omega^$ are non- $\approx_{\text{RK}}$ -equivalent, RK-minimal ultrafilters, then $p \otimes q$ satisfies $(*)$.*

Proof. Observe that p cannot be RK-minimal since $r <_{\text{RK}} p$. We only prove the necessity. Let $f, g \in {}^\omega\omega$ be such that $\bar{f}(p) = \bar{g}(p)$. If $\bar{f}(p) = \bar{g}(p) = k \in \omega$, then $f^{-1}(k) \cap g^{-1}(k) \in p$. Hence, we may assume that $\bar{f}(p) = \bar{g}(p) \in \omega^*$. We proceed by contradiction. Suppose that $\{n < \omega: f(n) = g(n)\} = \emptyset$. We consider three cases:

Case 1. $A = \{n < \omega: \bar{f}(e(n)) \in \omega\} \in r$. Then $\bar{f}(\bar{e}(r)) = \bar{f}(p) \approx_{\text{RK}} r$. So $\bar{f}(p) = \bar{g}(p)$ is RK-minimal; hence, it is a P -point of ω^* , because every RK-minimal ultrafilter is a P -point of ω^* . Since $\bar{g}(p) \in \text{Cl}_{\beta(\omega)}\{g(p_n): n < \omega\}$ and $\bar{g}(p)$ is a P -point of ω^* , either $\{n < \omega: \bar{g}(p_n) = \bar{g}(p)\} \in r$ or $\{n < \omega: \bar{g}(p_n) \in \omega\} \in r$. Since r is not $\approx_{\text{RK}}$ -equivalent to p_n and $\bar{g}(p_n) \leq_{\text{RK}} p_n$ for all $n < \omega$, $B = \{n < \omega: \bar{g}(p_n) \in \omega\} \in r$. Thus $\bar{f} \circ e|_{A \cap B}$ and $\bar{g} \circ e|_{A \cap B}$ are in ${}^{A \cap B}\omega$. Applying Lemma 2.1 and the assumption that r is RK-minimal, we have that $\{n < \omega: \bar{f}(e(n)) = \bar{g}(e(n))\} \in r$. Hence, there is $m < \omega$ such that $\bar{f}(p_m) = \bar{g}(p_m)$. Applying again Lemma 2.1, we obtain that $\{n < \omega: f(n) = g(n)\} \in p_m$, which is a contradiction.

Case 2. $\{n < \omega: \bar{f}(p_n) = \bar{f}(p)\} \in r$. Then $\bar{f}(p) = \bar{g}(p) \approx_{\text{RK}} p_m$ for some $m < \omega$, so $\bar{g}(p)$ is a P -point of ω^* . Hence, either $N = \{n < \omega: \bar{g}(p_n) = \bar{g}(p)\} \in r$ or $M = \{n < \omega: \bar{g}(p_n) \in \omega\} \in r$. If $N \in r$ then there is $k < \omega$ such that $\bar{f}(p_k) = \bar{g}(p_k)$, but this contradicts the hypothesis, by Lemma 2.1. If $M \in r$, then $\bar{g}(\bar{e}(r)) = \bar{g}(p) \approx_{\text{RK}} r$, which is impossible since, in this case, $\bar{g}(p) \approx_{\text{RK}} p_m$ for some $m < \omega$.

Case 3. $\{n < \omega: \bar{f}(p) \neq \bar{f}(p_n) \text{ and } \bar{f}(p_n) \in \omega^*\} \in r$. Since r is RK-minimal, by [C, Lemma 9.4], there is $A \in r$ such that $\bar{f} \circ e|_A$ is an embedding. For each $n \in A$, we have that $\bar{f}(p_n) = \bar{f}(e(n)) \approx_{\text{RK}} p_n = e(n)$. Since X is a set of P -points of ω^* , X is a (strongly) discrete subset of ω^* . Hence, e is an embedding. We need the following fact (see [Bo, Lemma 2.20]): Let $\phi, \psi: \omega \rightarrow \omega^*$ be embeddings and let $q \in \omega^*$. Then

$$\{n < \omega: \phi(n) \approx_{\text{RK}} \psi(n)\} \in q \Leftrightarrow \bar{\phi}(q) \approx_{\text{RK}} \bar{\psi}(q).$$

From this fact it follows that $\overline{f}(p) = \overline{f}(\overline{e}(r)) \approx_{\text{RK}} \overline{e}(r) = p$. By Lemma 1.1(1), there is $B \in p$ such that $f|_B$ is one-to-one. By Lemma 2.1, $\{n < \omega: f(n) = g(n)\} \in p$, but this contradicts our assumption.

2.3. Theorem. *If $\omega \leq \kappa \leq \alpha$ and $p \in U(\kappa) \subseteq \beta(\kappa) \subseteq \beta(\alpha)$, then*

$$|T_{\text{RK}}(p)| = |\kappa\alpha/p|.$$

Proof. By the definition of type, for every $q \in T_{\text{RK}}(p)$ there is a permutation σ_q of α such that $\overline{\sigma}_q(p) = q$. Define $f_q = \sigma_q \circ i$ for $q \in T_{\text{RK}}(p)$, where i is the inclusion map $i: \kappa \rightarrow \alpha$. Then, define $\Psi: T_{\text{RK}}(p) \rightarrow \kappa\alpha/p$ by $\Psi(q) = f_q/p$ for $q \in T_{\text{RK}}(p)$. We claim that Ψ is one-to-one. Indeed, if

$$\{\xi < \kappa: f_q(\xi) = f_r(\xi)\} = \kappa \cap \{\xi < \alpha: \sigma_q(\xi) = \sigma_r(\xi)\} \in p$$

for some $q, r \in T_{\text{RK}}(p)$, then $\{\xi < \alpha: \sigma_q(\xi) = \sigma_r(\xi)\} \in p$; hence, $\overline{\sigma}_q(p) = q = \overline{\sigma}_r(p) = r$. This shows that Ψ is one-to-one, so $|T_{\text{RK}}(p)| \leq |\kappa\alpha/p|$. Now, let $\{A_\xi: \xi < \alpha\}$ be a partition of α such that $|A_\xi| = \alpha$ for $\xi < \alpha$. Index A_ξ by $\{a(\xi, \eta): \eta < \alpha\}$ for $\xi < \alpha$. For each $f \in \kappa\alpha$, define $\phi_f \in \alpha\alpha$ by $\phi_f(\xi) = a(\xi, f(\xi))$, for $\xi < \kappa$, and if $\kappa < \alpha$, then ϕ_f is any bijection between $\alpha \setminus \kappa$ and $\alpha \setminus \{a(\xi, f(\xi)): \xi < \kappa\}$. It is evident that ϕ_f is one-to-one for each $f \in \kappa\alpha$. Set $p_f = \overline{\phi}_f(p)$ for $f \in \kappa\alpha$. By Lemma 1.1(1), we have that $p_f \in T_{\text{RK}}(p)$ for each $f \in \kappa\alpha$. Then, we define $\Phi: \kappa\alpha/p \rightarrow T_{\text{RK}}(p)$ by $\Phi(f/p) = p_f$, for $f \in \kappa\alpha$, and verify that Φ is well defined and one-to-one. In fact, if $f, g \in \kappa\alpha$ and $\{\xi < \kappa: f(\xi) = g(\xi)\} \in p$, then $\{\xi < \alpha: \phi_f(\xi) = \phi_g(\xi)\} \in p$; hence, $p_f = \overline{\phi}_f(p) = p_g = \overline{\phi}_g(p)$. Thus, Φ is well defined. Assume that $f, g \in \kappa\alpha$ satisfy $p_f = \Phi(f/p) = \overline{\phi}_f(p) = \Phi(g/p) = \overline{\phi}_g(p) = p_g$. By Lemma 2.1, $\{\xi < \kappa: \phi_f(\xi) = \phi_g(\xi)\} = \{\xi < \kappa: f(\xi) = g(\xi)\} \in p$; that is, $f \equiv_p g$. Thus, Φ is one-to-one, so $|T_{\text{RK}}(p)| \leq |\kappa\alpha/p| \leq |T_{\text{RK}}(p)|$.

3. COMFORT ORDER

In [G1, G2], W. W. Comfort defined an order on ω^* as a tool to study the relation among p -compact properties and some topological properties of ω^* . This section is devoted to investigate the properties of Comfort order for cardinals higher than ω . We begin with the definition of Comfort order.

3.1. Definition (Comfort). For $p, q \in \alpha^*$, we say $p \leq_c q$ if every q -compact space is p -compact.

It is evident that $\leq_c$ is a pre-order on α^* and $\leq_{\text{RK}} \subseteq \leq_c$. For $p, q \in \alpha^*$, we say $p \approx_c q$ if $p \leq_c q \leq_c p$ and the *Comfort type* of p is the set $T_c(p) = \{q \in \alpha^*: p \approx_c q\}$. It is evident that if $q \in T_c(p)$ then $T_{\text{RK}}(q) \subseteq T_c(p)$ (see [G1] or [G2]). Thus, each Comfort type of α^* can be filled out with types of α^* . This suggests the following:

3.2. Definition. For $p \in \alpha^*$, we define

$$c_p = |\{T_{\text{RK}}(q): q \in T_c(p)\}|.$$

The next easy theorem is basic in the study of Comfort order.

3.3. Theorem. For $p, q \in \alpha^*$, the following conditions are equivalent.

- (1) $p \leq_c q$.
- (2) $\beta_p(\alpha) \subseteq \beta_q(\alpha)$.
- (3) $p \in \beta_q(\alpha)$.
- (4) $\exists f \in {}^\alpha \beta_q(\alpha) (\bar{f}(q) = p \notin f[\alpha])$.
- (5) $\beta_q(\alpha)$ is p -compact.
- (6) $\beta_q(\alpha) \cap \alpha^*$ is p -compact.

3.4. Lemma. Let $p, q \in \alpha^*$. If $p \leq_c q$ then $\|p\| \leq \|q\|$. In particular, if $p \in U(\alpha)$ then $T_c(p) \subseteq U(\alpha)$.

Proof. Set $\gamma = \|q\|$. We may assume that $q \in U(\gamma)$. By Theorem 3.3 and the definition of $\beta_q(\omega)$, it suffices to show that the set $X = \{r \in \alpha^* : \|r\| \leq \|q\|\} \cup \alpha$ is q -compact. In fact, let $f \in {}^\gamma X$. For each $\xi < \gamma$, choose $A_\xi \in f(\xi)$ such that $|A_\xi| \leq \gamma$. Hence, we have that $f[\gamma] \subseteq \hat{A}$, where $A = \bigcup_{\xi < \gamma} A_\xi$, and so $\bar{f}(q) \in \hat{A}$. Since $|A| \leq |\bigcup_{\xi < \gamma} A_\xi| \leq \gamma$, $\bar{f}(q) \in X$. Thus, X is q -compact.

Now, we give some conditions equivalent to the definition of Comfort order involving the tensor product of ultrafilters and RK-order; the proof is completely similar to that of [G2, Corollary 2.6(4)].

3.5. Lemma. For $p, q \in \alpha^*$, the following statements are equivalent.

- (1) $p \leq_c q$.
- (2) $p \otimes q \approx_c q \otimes p \approx_c q$.
- (3) $\exists r \in T_c(q)(p \leq_{\text{RK}} r)$.

In addition, if $p, q \in T_c(r)$ then $p \otimes q \in T_c(r)$.

In [G1, G2], the author showed that $c_p = 2^\omega$ for each $p \in \omega^*$. Thus, it is natural to ask whether $c_p = 2^\alpha$ for $p \in U(\alpha)$. We will prove that if p is decomposable then $c_p = 2^\alpha$, and $c_p = \omega$ whenever $\alpha > \omega$ and $p \in U(\alpha)$ is RK-minimal (the existence of such a p makes α a measurable cardinal, because p must be α -complete). First, we prove some preliminary results.

3.6. Lemma. Let $p, q \in \alpha^*$. If $p \leq_c q$, then $\omega \leq c_p \leq c_q \leq 2^\alpha$.

Proof. It follows directly from Lemmas 1.1(4) and 3.5(2).

Notice from Lemma 3.6 that if $p \in U(\alpha)$ is countably incomplete then $2^\omega \leq c_p$. Moreover, we will show that $c_p = c_p^\omega$ whenever $p \in U(\alpha)$ is countably incomplete. We need the following lemmas (a proof of Lemma 3.7 is in [G1]).

3.7. Lemma. Let $\omega \leq \kappa \leq \alpha$ be a cardinal, $p \in U(\kappa)$, and $\{A_\xi : \xi < \alpha\}$ a partition of α . If $f, g \in {}^\kappa(\alpha^*)$ and $f(\xi), g(\xi) \in \hat{A}_\xi$, then $\bar{f}(p) \approx_{\text{RK}} \bar{g}(p)$ iff $\{\xi < \alpha : f(\xi) \approx_{\text{RK}} g(\xi)\} \in p$.

3.8. Lemma. If $p \in U(\alpha)$ and $\omega \leq \kappa \leq c_p$, then $|{}^\alpha \kappa / p| \leq c_p$.

Proof. Let $\{A_\xi : \xi < \alpha\}$ be a partition of α in subsets of cardinality α . For each $\xi < \alpha$, choose a subset $\{p(\xi, \zeta) : \zeta < \kappa\}$ of $T_c(p) \cap \hat{A}_\xi$ so that $p(\xi, \zeta)$ is not $\approx_{\text{RK}}$ -equivalent to $p(\xi, \eta)$ whenever $\zeta < \eta < \kappa$. Then, we define $\Phi : {}^\alpha \kappa / p \rightarrow T_c(p)$ by $\Phi(f/p) = \bar{\phi}_f(p) = p_f$ for $f \in {}^\alpha \kappa$, where $\phi_f : \alpha \rightarrow \alpha^*$ is defined by $\phi_f(\xi) = p(\xi, f(\xi))$ for $\xi < \alpha$. By Lemma 3.7, Φ is well defined

and

$$p_f \approx_{\text{RK}} p_g \quad \text{iff} \quad \{\xi < \alpha: p(\xi, f(\xi)) \approx_{\text{RK}} p(\xi, g(\xi))\} \in p \\ \text{iff} \quad \{\xi < \alpha: f(\xi) = g(\xi)\} \in p;$$

that is, $T_{\text{RK}}(p_f) \neq T_{\text{RK}}(p_g)$ whenever $f/p \neq g/p$ for $f, g \in {}^\alpha\kappa$. Observe that $p \leq_{\text{RK}} p_f$ for each $f \in {}^\alpha\kappa$. Since $T_c(p) \subseteq \beta_p(\alpha)$ and $\beta_p(\alpha)$ is p -compact, $p_f \in \beta_p(\alpha)$ for $f \in {}^\alpha\kappa$. By Theorem 3.3, we have that $p_f \leq_c p$; hence, $p_f \in T_c(p)$, for $f \in {}^\alpha\kappa$. Therefore, $|{}^\alpha\kappa/p| \leq c_p$.

The proofs of the following results are available in [CN].

3.9. Lemma. *Let $p \in U(\alpha)$. Then:*

- (1) (Keisler) *if κ is an arbitrary cardinal, then $|{}^\alpha(\kappa^{<\alpha})/p| = \kappa^\alpha$;*
- (2) (Keisler) *if p is countably incomplete and κ is a cardinal, then $|{}^\alpha\kappa/p|^\omega = |{}^\alpha\kappa/p|$;*
- (3) (Frayne-Morel-Scott) $\alpha < |{}^\alpha\alpha/p|$.

3.10. Lemma. *Let $f, g \in {}^\alpha(\alpha^*)$ and $p \in \alpha^*$. If g is a strong embedding and $\{\xi < \alpha: f(\xi) \leq_{\text{RK}} g(\xi)\} \in p$, then $\bar{f}(p) \leq_{\text{RK}} \bar{g}(p)$.*

Proof. Let $\{A_\xi: \xi < \alpha\}$ be a partition of α such that $g(\xi) \in \hat{A}_\xi$ for each $\xi < \alpha$. Without loss of generality, we may assume that $f(\xi) \leq_{\text{RK}} g(\xi)$ for all $\xi < \alpha$. Choose $h_\xi: A_\xi \rightarrow \alpha$ so that $\bar{h}_\xi(g(\xi)) = f(\xi)$, for each $\xi < \alpha$, and define $h = \bigcup_{\xi < \alpha} h_\xi$. Then we have that $\bar{h}(\bar{g}(p)) = \bar{f}(p)$, as required.

3.11. Theorem. *Let $p \in U(\alpha)$. Then:*

- (1) *if $\alpha \leq c_p$, then $\alpha < c_p \leq 2^\alpha$;*
- (2) *if p is countably incomplete, then $2^\omega \leq c_p = c_p^\omega \leq 2^\alpha$;*
- (3) *if p is decomposable, then $c_p = 2^\alpha$;*
- (4) *if p is regular, then $c_p = 2^\alpha$;*
- (5) *if $p \in U(\aleph_n)$, then $c_p = 2^{\aleph_n}$ for $n < \omega$.*

Proof. (1) follows from Lemmas 3.8 and 3.9(3).

(2) Since p is countably, incomplete, there is $r \in \omega^*$ such that $r \leq_{\text{RK}} p$. According to Lemmas 3.6, 3.8, 3.9(2) and the fact that $c_r = 2^\omega$ (see [G1] or [G2]), we obtain that

$$2^\omega \leq c_r \leq c_p \leq c_p^\omega \leq |{}^\alpha c_p/p|^\omega = |{}^\alpha c_p/p| \leq c_p \leq 2^\alpha.$$

(3) First, we show that there is an increasing chain $\{p_\gamma: \omega \leq \gamma \leq \alpha\}$ in the RK-order such that $p_\gamma \in U(\gamma) \cap \beta_p(\alpha)$, for each $\omega \leq \gamma \leq \alpha$. In fact, since p is decomposable, for each $\omega \leq \gamma \leq \alpha$ there is $q_\gamma \in U(\gamma)$ such that $q_\gamma \leq_{\text{RK}} p$ and $q_\alpha = p$. Observe that $\beta_p(\alpha)$ is q_γ -compact for each $\omega \leq \gamma \leq \alpha$ (by Theorem 3.3). Now, we proceed by transfinite induction. Let $p_\omega = q_\omega$, and assume that p_γ has been defined for $\omega \leq \gamma < \lambda \leq \alpha$ so that:

- (1) $p_\gamma \in U(\gamma) \cap \beta_p(\alpha)$ for $\omega \leq \gamma < \lambda$;
- (2) $p_\delta \leq_{\text{RK}} p_\gamma$ for cardinals $\omega \leq \delta < \gamma < \lambda$.

If $\lambda = \gamma^+$, then we define $p_\lambda = q_\lambda \otimes p_\gamma$ and, by Lemmas 1.1(3) and 3.5, p_λ satisfies (1) and (2). Suppose that λ is a limit cardinal. Let $\{A_\xi: \xi < \lambda\}$ be a partition of λ in subsets of cardinality λ . Define $f \in {}^\lambda\beta(\lambda)$ such that $f(\xi) \in \hat{A}_\xi \cap T_{\text{RK}}(p_{|\xi|})$, for each $\omega \leq \xi < \lambda$, and set $p_\lambda = \bar{f}(q_\lambda)$. Notice that, for each $\omega \leq \gamma < \lambda$, $p_\gamma \leq_{\text{RK}} f(\xi)$ for all ξ such that $\gamma^+ \leq \xi < \lambda$. By Lemmas

1.1(2), 3.10 and the q_λ -compactness of $\beta_p(\alpha)$, we have that $p_\gamma \leq_{\text{RK}} q_\lambda \otimes p_\gamma \leq_{\text{RK}} \bar{f}(q_\lambda) = p_\lambda$ and $p_\lambda \in U(\lambda) \cap \beta_p(\alpha)$.

Now, consider the chain $\{p_\gamma: \omega \leq \gamma \leq \alpha\}$ defined above. Without loss of generality, we may suppose that $p_\alpha = p$. We will prove, by induction, that $c_{p_\gamma} = 2^\gamma$ for each $\omega \leq \gamma \leq \alpha$. It is a fact that $c_{p_\omega} = 2^\omega$, since $p_\omega \in \omega^*$. Assume $c_{p_\gamma} = 2^\gamma$ for all $\omega \leq \gamma < \lambda \leq \alpha$. Since $p_\gamma \leq_{\text{RK}} p_\lambda$, by Lemma 3.6, we have that $2^\gamma = c_{p_\gamma} \leq c_{p_\lambda}$, for $\omega \leq \gamma < \lambda$. This implies that $2^{<\lambda} \leq c_{p_\lambda}$. From Lemmas 3.8 and 3.9(1) it follows that $2^\lambda = |\lambda(2^{<\lambda})/p_\lambda| \leq c_{p_\lambda} \leq 2^\lambda$. Therefore, $c_p = 2^\alpha$.

Clauses (4) and (5) follow from (3) and a theorem due to G. V. Čudnovskij and D. V. Čudnovskij [CC] and Kunen and Prikry [KP] that every regular ultrafilter on α and every $q \in U(\aleph_n)$, for each $n < \omega$, are decomposable (see [BS]).

Clause (5) of Theorem 3.11 leads us to ask:

3.12. **Question.** Is $c_p = 2^{\aleph_\omega}$ for all $p \in U(\aleph_\omega)$?

Donder [D] has shown that in the core model K every ultrafilter is regular. Hence, by Theorem 3.11(3), $K \models \forall p \in U(\alpha) (c_p = 2^\alpha)$. Nevertheless, we will prove that, if $p \in U(\alpha)$ is RK-minimal and $\alpha > \omega$, then $c_p = \omega$. The following lemmas are needed.

3.13. **Lemma.** If $p, q \in U(\alpha)$ are α -complete, then $p \otimes q$ is α -complete.

Proof. First, we give some notation. For $A \subseteq \alpha \times \alpha$, $\xi < \alpha$, and $r \in \alpha^*$, we let $A(\xi) = \{\zeta < \alpha: (\xi, \zeta) \in A\}$ and $S(A, r) = \{\xi < \alpha: A(\xi) \in r\}$. It is evident that $A \in p \otimes q \Leftrightarrow S(A, q) \in p$. Let $\{A_\eta: \eta < \kappa\} \subseteq p \otimes q$ for $\kappa < \alpha$, and set $A = \bigcap_{\eta < \kappa} A_\eta$. Clearly, $A(\xi) = \bigcap_{\eta < \kappa} A_\eta(\xi)$ for $\xi < \alpha$. It suffices to show that $S(A, q) = \bigcap_{\eta < \kappa} S(A_\eta, q)$, because p is α -complete. Indeed, $\xi \in S(A, q) \Leftrightarrow A(\xi) = \bigcap_{\eta < \kappa} A_\eta(\xi) \in q \Leftrightarrow A_\eta \in q$ for all $\eta < \kappa$ [since q is α -complete] $\Leftrightarrow \xi \in S(A_\eta, q)$ for all $\eta < \kappa \Leftrightarrow \xi \in \bigcap_{\eta < \kappa} S(A_\eta, q)$.

We omit the proof of the next easy lemma.

3.14. **Lemma.** If $p \in U(\alpha)$ is α -complete, then $p \notin D^-$ for each $D \subseteq \alpha^* \setminus \{p\}$ with $|D| < \alpha$.

3.15. **Lemma.** If $p \in U(\alpha)$ is RK-minimal, $1 \leq n < \omega$, and $f \in {}^\alpha[T(p)^n]$ satisfies $\bar{f}(p) = q \notin f[\alpha]$, then $q \approx_{\text{RK}} p^{n+1}$.

Proof. Since p is RK-minimal, p is a P_α -point of $U(\alpha)$ and α -complete (see [CN]). By virtue of Lemma 1.1(2), it is enough to show that there is $A \in p$ such that $f|_A$ is a strong embedding. In fact, for each $\xi < \alpha$ let $C_\xi = f^{-1}(\{f(\xi)\})$. Then $C_\xi \notin p$ for all $\xi < \alpha$. Since p is RK-minimal, there is $C \in p$ such that $|C \cap C_\xi| \leq 1$ for each $\xi < \alpha$ (by [CN, Theorem 9.6]). Namely, $f|_C$ is one-to-one. Hence, without loss of generality, we may assume that f is one-to-one. For every $\xi < \alpha$, choose $A_\xi \in f(\xi)$ such that $A_\xi \notin q$. Then $B_\xi = \{\zeta < \alpha: f(\zeta) \notin \hat{A}_\xi\} \in p$ for $\xi < \alpha$. Since p is a P_α -point, there is $B \in p$ such that $|B \setminus B_\xi| < \alpha$ for $\xi < \alpha$. By Lemma 3.13, $f(\xi)$ is α -complete for $\xi < \alpha$; hence, $f(\xi) \notin f[(B \setminus \{\xi\}) \setminus B_\xi]^-$ (by Lemma 3.14). Define $U_\xi = \hat{A}_\xi \setminus f[(B \setminus \{\xi\}) \setminus B_\xi]^-$ for each $\xi \in B$. Then $U_\xi \cap f[B] = \{f(\xi)\}$ for $\xi \in B$. Thus, $f|_B$ is one-to-one and discrete. We may suppose that $B = \alpha$. Now, we

show that f is a strong embedding. By induction, we will define a suitable partition of α as follows.

Let $R_0 \in f(0)$ such that $R_0 \notin q$ and $R_0 \cap f[\alpha] = \{f(0)\}$. Assume that we have defined $R_\xi \in f(\xi)$ for $\xi < \eta < \alpha$ so that:

- (1) $R_\zeta \cap R_\xi = \emptyset$ for $\zeta < \xi < \eta$;
- (2) $R_\xi \notin q$ for $\xi < \eta$;
- (3) $R_\xi \cap f[\alpha] = \{f(\xi)\}$ for $\xi < \eta$.

Set $R = \bigcup_{\xi < \eta} R_\xi$. By Lemma 3.13, we have that $f(\zeta)$ is α -complete, for all ζ such that $\eta \leq \zeta < \alpha$. Hence, $f[[\eta, \alpha)] \cap \hat{R} = \emptyset$, since $\bigcap_{\xi < \eta} (\alpha \setminus R_\xi) \in f(\zeta)$ for each ζ such that $\eta \leq \zeta < \alpha$. Thus, $f[[\eta, \alpha)] \subseteq (\alpha \setminus R)^\wedge$. Since $f[[\eta, \alpha)]$ is a discrete subset of $U(\alpha)$, we can find $R_\eta \in f(\eta)$ such that $R_\eta \cap f[\alpha] = \{f(\eta)\}$, $R_\eta \subseteq \alpha \setminus R$, and $R_\eta \notin q$. Thus, $\{R_\xi : \xi < \alpha\}$ witnesses that f is a strong embedding.

As a direct consequence of Lemma 3.15 we have:

3.16. Theorem. *If $\omega < \alpha$ and $p \in U(\alpha)$ is RK-minimal, then $c_p = \omega$.*

Proof. It suffices to show that $X = (\bigcup \{T_{RK}(p)^n : n < \omega\}) \cup \alpha$ is p -compact; hence, $T_c(p) = \bigcup \{T_{RK}(p)^n : n < \omega\}$. Indeed, let $f \in {}^\alpha X$ with $q = \bar{f}(p) \notin f[\alpha]$. If $\{\xi < \alpha : f(\xi) \in \alpha\} \in p$, then $q \in T_{RK}(p)$, since p is RK-minimal. Hence, we may assume that $f[\alpha] \subseteq \bigcup \{T_{RK}(p)^n : n < \omega\}$. Define $A_n = \{\xi < \alpha : f(\xi) \in T_{RK}(p)^n\}$ for $1 \leq n < \omega$. Since p is ω_1 -complete, there is $1 \leq n < \omega$ such that $A_n \in p$. Hence, without loss of generality, we may suppose that $f[\alpha] \subseteq T_{RK}(p)^n$. By Lemma 3.15, we have that $q \in T_{RK}(p)^{n+1} \subseteq X$. Therefore, X is p -compact.

It follows from Lemma 3.15, Theorem 3.11(2), and Theorem 9.6 of [CN] that a cardinal α is measurable if and only if there is $p \in U(\alpha)$ such that $c_p < 2^\omega$. It should be mentioned that if α is measurable then there is $p \in U(\alpha)$ such that p is α -complete and $c_p = 2^\alpha$; indeed, by [CN, Theorem 10.9], we can find $p \in U(\alpha)$ so that p is α -complete and p has 2^α many Rudin-Keisler type predecessors, and we now apply Lemmas 1.1(4) and 3.5(2).

In the next theorem, we will show that if α is a strong limit and $p \in U(\alpha)$ is indecomposable, then every $q \in T_c(p)$ is indecomposable. Prikry [P] proved that if $p \in U(\alpha)$ is α -complete then there is a generic extension in which any ultrafilter extending p is indecomposable and α is a strong limit with $cf(\alpha) = \omega$. Woodin (unpublished) showed that we can collapse a measurable cardinal to $\aleph_\omega$ so that, in the generic extension, $\aleph_\omega$ is a strong limit and there is an indecomposable ultrafilter over $\aleph_\omega$.

3.17. Theorem. *Let α be a strong limit and $p \in U(\alpha)$ indecomposable. If $q \in U(\alpha)$ and $q \leq_c p$, then q is indecomposable.*

Proof. We will verify that the space $X = \alpha \cup [\beta_p(\alpha) \cap (N \cup U(\alpha))]$ is p -compact, where $N = \{q \in \beta_p(\alpha) : \|q\| = \omega\}$. Indeed, let $f \in {}^\alpha X$ such that $q = \bar{f}(p) \notin f[\alpha]$. Since $\beta_p(\alpha) \cap U(\alpha)$ is p -compact, we may assume that $f[\alpha] \subseteq \alpha \cup (\beta_p(\alpha) \cap N)$. If $\{\xi < \alpha : f(\xi) \in \alpha\} \in p$ then $q \leq_{RK} p$; hence, $q \in \beta_p(\alpha) \cap (N \cup U(\alpha))$, since p is indecomposable. Then, we may suppose that $f[\alpha] \subseteq \beta_p(\alpha) \cap N$. Assume that $q \in N(\alpha)$, and fix $A \in q$ with $|A| < \alpha$. Since α is a strong limit, $|\hat{A}| < \alpha$. We have that $A_\xi = f^{-1}(f(\{\xi\})) \notin p$ for $\xi < \alpha$,

and, since $\{\xi < \alpha: f(\xi) \in \widehat{A}\} \in p$, it follows that $B = \bigcup \{A_\xi: f(\xi) \in \widehat{A}\} \in p$. Consider the function $g \in {}^B\alpha$ defined by $g^{-1}(\{\xi\}) = A_\xi$ for each $\xi < \alpha$ with $f(\xi) \in \widehat{A}$. Since p is indecomposable, $\overline{g}(p) \leq_{\text{RK}} p$, and $A_\xi \notin p$ for $\xi < \alpha$, there is $\{f(\xi_n): n < \omega\} \subseteq \widehat{A}$ such that $\bigcup_{n < \omega} A_{\xi_n} \in p$; hence, $q \in \text{Cl}_{\beta(\alpha)}\{f(\xi_n): n < \omega\}$. For each $n < \omega$, choose $C_n \in f(\xi_n)$ such that $|C_n| = \omega$, since $f[\alpha] \subseteq N$. Then, we have that $q \in \widehat{C}$ where $C = \bigcup_{n < \omega} C_n$ and $|C| = \omega$. Therefore, $q \in \beta_p(\alpha) \cap N$. This proves that X is p -compact. Hence, the conclusion follows from the fact that $\leq_{\text{RK}} \subseteq \leq_c$ and Theorem 3.3.

Recall that a space X is *initially α -compact* if every open cover $\mathcal{U}$ of X with $|\mathcal{U}| \leq \alpha$ has a finite subcover. In [G3, Corollary 2.15] we have improved a theorem of Saks by establishing that X^γ is initially α -compact for each cardinal γ iff X is p -compact for some decomposable $p \in U(\alpha)$. In particular, we have

3.18. Theorem. *For $p \in U(\alpha)$, the following conditions are equivalent.*

- (1) $\beta_p(\alpha)$ is initially α -compact;
- (2) $\beta_p(\alpha) \cap U(\kappa) \neq \emptyset$ for all cardinal $\omega \leq \kappa < \alpha$;
- (3) $\beta_p(\alpha) \cap U(\kappa) \neq \emptyset$ for all regular κ with $\omega \leq \kappa < \alpha$;
- (4) there is $q \in T_c(p)$ decomposable.

It follows from Theorem 3.17 that if α is a strong limit and $p \in U(\alpha)$ is indecomposable, then $\beta_p(\alpha)$ is not initially α -compact. Thus, the condition decomposable in Corollary 2.15 in [G3] is essential. On the other hand, Donder's result implies that $K \models \forall p \in U(\alpha) (\beta_p(\alpha) \text{ is initially } \alpha\text{-compact})$ and $K \models \text{a space } X \text{ is } p\text{-compact for } p \in U(\alpha) \Leftrightarrow X^\gamma \text{ is initially } \alpha\text{-compact for each cardinal } \gamma$.

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INSTITUTO DE MATEMÁTICAS, CIUDAD UNIVERSITARIA (UNAM), MÉXICO, D.F. 04510, MÉXICO
E-mail address: garcia@redvax1.dgsca.unam.mx