A WEIGHTED INEQUALITY FOR THE TAKEYA MAXIMAL OPERATOR

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(Communicated by J. Marshall Ash)

ABSTRACT. Let $\mathcal{K}_\delta$ be the Takeya Maximal Operator defined as the supremum of averages over parallelepipeds of eccentricity δ . We show that $\mathcal{K}_\delta$ satisfies $\|\mathcal{K}_\delta f\|_{L^p(\omega)} \leq C_{n,p} (1/\delta)^{n/p-1} (\log(1/\delta))^{\alpha_n} \|f\|_{L^p(\mathcal{K}_\delta \omega)}$ for all $p \leq (n+1)/2$ with some constants $C_{n,p}$, α_n , independent of f and the weight ω .

For $0 < \delta < 1$ and $f \in L^1_{\text{loc}}(\mathbf{R}^n)$, we define the Takeya Maximal Function of eccentricity δ

$$\mathcal{K}_\delta f(x) = \sup \frac{1}{|R_x|} \int_{R_x} |f(y)| dy,$$

where the 'sup' is taken over all rectangular parallelepipeds in $\mathbf{R}^n$ homothetic to $[0, \delta]^{n-1} \times [0, 1]$ of arbitrary direction and containing x .

It is conjectured that $\mathcal{K}_\delta$ is bounded on $L^n(\mathbf{R}^n)$ with a constant given by

$$(1) \quad C_n \left(\log \frac{1}{\delta} \right)^{\alpha_n}$$

for some $C_n, \alpha_n > 0$.

If the conjecture were true, we would obtain, by interpolation, the following L^p estimates:

$$(2) \quad \|\mathcal{K}_\delta f\|_{L^p(\mathbf{R}^n)} \leq \begin{cases} C_n \left(\log \frac{1}{\delta} \right)^{\alpha_n} \|f\|_{L^p(\mathbf{R}^n)} & \text{for } p \geq n, \\ C_{n,p} \left(\frac{1}{\delta} \right)^{n/p-1} \left(\log \frac{1}{\delta} \right)^{\alpha_n} \|f\|_{L^p(\mathbf{R}^n)} & \text{for } 1 < p < n. \end{cases}$$

To see it, note that the Takeya operator is bounded on $L^\infty(\mathbf{R}^n)$ with constant 1 and is of weak-type 1 with constant $C_n \left(\frac{1}{\delta} \right)^{n-1}$. (The latter follows from the classical result for the Hardy-Littlewood maximal operator.)

For $n = 2$ the conjecture is known to be true (see [C]). More recently, the following double-weight L^2 estimate has been proved

$$(3) \quad \|\mathcal{K}_\delta f\|_{L^2(\omega)} \leq C \left(\log \frac{1}{\delta} \right)^\alpha \|f\|_{L^2(\mathcal{K}_\delta \omega)}$$

for all weights ω in $\mathbf{R}^2$ (see [MS]).

Received by the editors January 6, 1992 and, in revised form, July 8, 1992.

1991 *Mathematics Subject Classification*. Primary 42B25.

Key words and phrases. Maximal functions, weighted inequalities.

In higher dimensions the question is still open. Christ, Duoandikoetxea, and Rubio de Francia [CDR] established (2) for $p \leq \frac{n+1}{2}$. Recently, Bourgain [B] proved that there is an exponent $p_n > \frac{n+1}{2}$ so that (2) holds for $p = p_n$.

In this note we establish the following weighted version of the result in [CDR]

Theorem 1. *For $n \geq 3$ and $1 < p \leq \frac{n+1}{2}$ there are constants $C_{n,p}$, $\alpha_n > 0$ such that, for every $1 > \delta > 0$ and every positive weight ω , we have*

$$(4) \quad \|\mathcal{H}_\delta f\|_{L^p(\omega)} \leq C_{n,p} \left(\frac{1}{\delta}\right)^{n/p-1} \left(\log \frac{1}{\delta}\right)^{\alpha_n} \|f\|_{L^p(\mathcal{H}_\delta \omega)}.$$

Let us consider another maximal function

$$\mathcal{M}_\delta f(x) = \sup \frac{1}{|R_x|} \int_{R_x} |f(y)| dy,$$

where the ‘sup’ is taken over rectangular parallelepipeds in $\mathbf{R}^n$, containing x , regardless of orientation, which have one side of length 1 and $n-1$ sides, each of length δ .

By interpolation and a ‘sieve’ argument, which will be discussed later, Theorem 1 is a consequence of the following restricted weak-type estimate:

Theorem 2. *There is a constant $C_n > 0$ such that*

$$(5) \quad \omega(\{x : \mathcal{M}_\delta(\chi_A)(x) > \sigma\}) \leq C_n \sigma^{-(n+1)/2} \delta^{-(n-1)/2} \mathcal{H}_\delta \omega(A),$$

for all $A \subset B(0, 1)$, $\sigma > 0$, $1 > \delta > 0$.

Proof of Theorem 2. For every $y \in \{x : \mathcal{M}_\delta(\chi_A)(x) > \sigma\}$ we select a parallelepiped, R_y , as in the definition of the maximal function and such that $|R_y \cap A| > \sigma |R_y|$. Following Bourgain [B], for $x \in A$ we define the ‘bush’ centered at x as $\mathcal{B}_x = \bigcup_{x \in R_y} R_y$. We want to show that there is a point $x_o \in A$ such that

$$\omega(\mathcal{B}_{x_o}) \geq c_n \sigma \delta^{n-1} \frac{\omega(\{x : \mathcal{M}_\delta(\chi_A)(x) > \sigma\})}{|A|}.$$

To see this, observe that

$$\begin{aligned} \int_A \omega(\mathcal{B}_x) dx &= \int_A \int_{\mathbf{R}^n} \chi_{\mathcal{B}_x}(y) \omega(y) dy dx \\ &\geq \int_{\{x : \mathcal{M}_\delta(\chi_A)(x) > \sigma\}} \omega(y) \int_A \chi_{\mathcal{B}_x}(y) dx dy. \end{aligned}$$

Now, note that, if $\mathcal{M}_\delta(\chi_A)(y) > \sigma$ and $x \in R_y$, then $y \in \mathcal{B}_x$, so that $\chi_{\mathcal{B}_x}(y) \geq \chi_{R_y}(x)$. Hence,

$$\begin{aligned} \int_A \omega(\mathcal{B}_x) dx &\geq \int_{\{x : \mathcal{M}_\delta(\chi_A)(x) > \sigma\}} \omega(y) \int_A \chi_{R_y}(x) dx dy \\ &\geq \int_{\{x : \mathcal{M}_\delta(\chi_A)(x) > \sigma\}} \omega(y) |R_y \cap A| dy \\ &\geq \int_{\{x : \mathcal{M}_\delta(\chi_A)(x) > \sigma\}} \omega(y) c_n \sigma \delta^{n-1} dy \\ &= c_n \sigma \delta^{n-1} \omega(\{x : \mathcal{M}_\delta(\chi_A)(x) > \sigma\}), \end{aligned}$$

and we obtain what we wanted.

On the other hand, suppose we had a set $\mathcal{E} \subset \{x : \mathcal{M}_\delta(\chi_A)(x) > \sigma\}$ and a point y_o verifying

- (a) $y_o \in \bigcap_{y \in \mathcal{E}} R_y$,
 (b) $\text{ang}(R_y, R_z) \geq 10\delta/\sigma$ if $y, z \in \mathcal{E}$, $y \neq z$.

(By $\text{ang}(R_y, R_z)$ we denote the angle between the main axes of R_y and R_z .)

Then, we could define $\mathcal{B} = \bigcup_{y \in \mathcal{E}} 3R_y$, a star-like set with respect to y_o , which would satisfy

$$\begin{aligned} \omega(\mathcal{B}) &\leq \sum_{y \in \mathcal{E}} \omega(3R_y) = \sum \left(\frac{1}{|3R_y|} \int_{3R_y} \omega \right) 3^n |R_y| \\ &\leq c_n \sum |R_y| \left(\inf_{R_y} \mathcal{H}_\delta \omega \right). \end{aligned}$$

Recall $|R_y \cap A| > \sigma |R_y|$. Moreover, $|R_y \cap A \cap B(y_o, \frac{\sigma}{3})| \leq |R_y \cap B(y_o, \frac{\sigma}{3})| \leq \frac{2\sigma}{3} |R_y|$. ($R_y \cap B(y_o, \frac{\sigma}{3})$ is contained in a parallelepiped with one side of length $\frac{2\sigma}{3}$ and $n-1$ sides of length δ .) Therefore, $|R_y \cap A \setminus B(y_o, \frac{\sigma}{3})| > \frac{\sigma}{3} |R_y|$, and then

$$\begin{aligned} \omega(\mathcal{B}) &\leq c_n \sum \frac{|R_y \cap A \setminus B(y_o, \sigma/3)|}{\sigma} \left(\inf_{R_y} \mathcal{H}_\delta \omega \right) \\ &\leq \sum \frac{c_n}{\sigma} \mathcal{H}_\delta \omega \left(R_y \cap A \setminus B(y_o, \frac{\sigma}{3}) \right) \leq \frac{c_n}{\sigma} \mathcal{H}_\delta \omega(\mathcal{B} \cap A). \end{aligned}$$

(The last inequality follows from the fact that the sets $\{R_y \cap A \setminus B(y_o, \frac{\sigma}{3})\}_{y \in \mathcal{E}}$ are mutually disjoint.)

Let us define, for the point x_o whose existence was proved above, the set

$$\mathcal{E}_{x_o} = \{y : x_o \in R_y\}.$$

Now, a moment's thought will convince us that we can choose a subset $\mathcal{E} \subset \mathcal{E}_{x_o}$ satisfying (a), with respect to x_o , and (b), such that

$$\omega \left(\bigcup_{y \in \mathcal{E}} 3R_y \right) \geq c_n \omega(\mathcal{B}_{x_o}) \sigma^{n-1}.$$

Then, from the construction of $\mathcal{B}_{x_o}$, the definition of $\mathcal{E}$, and the above computation on the ω -measure of $\mathcal{B} = \bigcup \{3R_y : y \in \mathcal{E}\}$, we have

$$\begin{aligned} \frac{c_n \sigma \delta^{n-1} \omega(\{x : \mathcal{M}_\delta(\chi_A)(x) > \sigma\})}{|A|} &\leq \omega(\mathcal{B}_{x_o}) \leq c_n \sigma^{1-n} \omega \left(\bigcup_{y \in \mathcal{E}} 3R_y \right) \\ &\leq c_n \sigma^{-n} \mathcal{H}_\delta \omega(A \cap \mathcal{B}) \leq c_n \sigma^{-n} \mathcal{H}_\delta \omega(A). \end{aligned}$$

Finally, if $x \in B(0, 2)$, then $\mathcal{K}_\delta \omega(x) \geq c_n \omega(B(0, 2))$. We deduce

$$\mathcal{K}_\delta \omega(A) \geq c_n \omega(B(0, 2))|A|,$$

and, therefore,

$$\begin{aligned} & \frac{c_n \sigma \delta^{n-1} \omega(\{x : \mathcal{M}_\delta(\chi_A)(x) > \sigma\})^2}{\mathcal{K}_\delta \omega(A)} \\ & \leq \frac{c_n \sigma \delta^{n-1} \omega(\{x : \mathcal{M}_\delta(\chi_A)(x) > \sigma\}) \omega(B(0, 2))}{\mathcal{K}_\delta \omega(A)} \\ & \leq \frac{c_n \sigma \delta^{n-1} \omega(\{x : \mathcal{M}_\delta(\chi_A)(x) > \sigma\})}{|A|} \leq c_n \sigma^{-n} \mathcal{K}_\delta \omega(A). \end{aligned}$$

This finishes with the proof of Theorem 2.

Proof of Theorem 1. Estimate (4) follows from Theorem 2, by standard arguments. First, we prove a restricted weak-type inequality (analogous to (5)) for $\mathcal{K}_\delta$:

$$\omega(\{x : \mathcal{K}_\delta(\chi_A)(x) > \sigma\}) \leq C_n \sigma^{-(n+1)/2} \delta^{-(n-1)/2} (\log \frac{1}{\delta})^{\beta_n} \mathcal{K}_\delta \omega(A).$$

In order to do that, we select some suitable families of parallelepipeds from the level set $\{\mathcal{K}_\delta(\chi_A) > \sigma\}$ and decompose A into disjoint subsets associated to the families. Each parallelepiped of a family belongs to one of, essentially, $\log \frac{1}{\delta}$ level sets associated to $\mathcal{M}_\delta$ and the corresponding subset. (This ‘sieve’ argument can be seen in [C] and [MS].)

Last of all, we use a three-point interpolation lemma as in [C] or, more directly, as in Proposition 5 in [CHS]. This will give us the strong type estimate (4).

Remark. Define

$$\mathcal{M}_\delta^6 f(x) = \sup \frac{1}{|R_x|} \int_{R_x} |f(y)| dy,$$

where the ‘sup’ is taken over rectangles in $\mathbf{R}^n$ congruent with $[0, 6\delta]^{n-1} \times [0, 6]$, of arbitrary direction and centered at x . Repeating carefully the proof of Theorem 2, one can obtain the sharper estimate

$$\omega(\{x : \mathcal{M}_\delta(\chi_A)(x) > \sigma\}) \leq C_n \sigma^{-(n+1)/2} \delta^{-(n-1)/2} \mathcal{M}_\delta^6 \omega(A)$$

for all $A \subset B(0, 1)$.

ACKNOWLEDGMENT

I would like to thank my adviser, F. Soria, for his help and friendship.

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